An axiomatic characterization of the Theil inequality ordering *

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Abstract

We identify an ordinal decomposability property and use it, along with other ordinal axioms, to characterize the Theil inequality ordering.

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1 Introduction

Economists have long been interested in income inequality. Typical issues include the evolution of income inequality over time in some particular region, the differences in income inequality across different regions, the effect of various policies on income inequality, and conversely, the effect of income inequality on various economic variables.[1] In order to address these and other similar questions one must first be able to measure income inequality, which is not a straightforward task.

The literature on income inequality measurement offers a plethora of inequality indices but the extent to which they are appropriate is not at all obvious. To compare the performance of different indices one may apply them to various distributions and check whether or not they contradict one's intuitions about inequality. For instance, they may be applied to two income distributions, one of which is believed to be more unequal than the other, and all those indices that contradict our subjective judgment may be discarded. Although this method may seem reasonable, it may not be very reliable, as discarding indices based on intuition is not the best scientific practice. Just as optical illusions may induce us to believe that one object is longer than another one while they are actually of equal length, so a false impression may induce us to believe that one income distribution is more unequal than another one, while in fact their level of inequality is the same.

Another, more cautious, way to evaluate inequality measures is to consider their properties at a more abstract level. We could compile a list of simple properties that a reasonable inequality measure should satisfy and then check which inequality measures do actually satisfy them. This method allows us to compare different indices in terms of the differential properties they do and do not satisfy, and has been successfully applied in the characterization of families of Gini-type indices, the Theil index, and the family of generalized entropy indices, among many others. In particular, Bourguignon [7] and Foster [15] have shown that

[1] See Goldberg and Pavcnik [17] for a recent survey on the effect of globalization on income inequality in developing countries, and Helpman et al. [18] for a theoretical analysis of the effect of trade liberalization on income inequality.

the Theil index of income inequality is the only index that satisfies several basic axioms as well as a simple decomposability property, known as Theil-Decomposability. In an important paper Shorrocks [24] shows that the generalized entropy indices are the unique indices (up to a monotone transformation) that are consistent with the Lorenz partial order and satisfy the ordinal property of aggregativity. Later on, Shorrocks [25] showed that aggregativity can be replaced by the ordinal axiom of subgroup consistency.

Some properties of inequality indices are uncontroversial, to the extent that they are considered to be the defining properties of the bare concept of inequality measure. One example is the Pigou-Dalton principle of transfers, which postulates that the transfer of income from a rich individual to a poorer one decreases inequality as long as the poor individual does not become richer than the rich one. Given other basic properties, this axiom is equivalent to requiring that the order be consistent with the Lorenz criterion. Other axioms, though very convenient, are less uncontroversial. For example, some require the inequality index to be decomposable in a particular way. Specifically, given any partition of a society into two subsocieties, they require that the overall inequality be decomposable into the inequality between these subsocieties, and the inequality within them. Though useful in applications, this decomposability is not at all a defining property of an inequality index. In fact, there are well-known inequality indices that are not decomposable.

It is important to bear in mind that some axioms are ordinal in nature, while others are cardinal. Ordinal axioms impose restrictions on how different income distributions are ranked. The Pigou-Dalton principle of transfers, for instance, is an ordinal property in that it compares two particular distributions and tells us which one is more unequal. It does not, however, relate to the magnitude of the inequality difference. Cardinal axioms, on the other hand, impose restrictions on the functional form of the index that is used to measure inequality. The decomposability property that Bourguignon [7] and Foster [15] use to characterize the Theil index is cardinal, since it requires that the total inequality of a region be a weighted sum of the inequalities of its subregions and the inequality between these subregions. This property is lost if we apply a non-linear monotonic transformation to the index.

In this paper we strip the decomposability property used by Bourguignon [7] and Foster [15] of all its cardinal content, and retain only its ordinal content. In particular, we identify an ordinal and meaningful decomposability property which is weaker than Theil Decomposability. This ordinal decomposability property states the following. Suppose we have two societies S and S' with the same total income but not necessarily the same population size. Identify a subsociety in each society with the same population size, n, and the same total income, y. The ranking of the two societies, S and S', in terms of income inequality should be independent of the way we distribute the total income y among the n members of the subpopulation. We use this property, along with other well-known ordinal properties, to characterize the Theil ordering of income inequality.

The rest of the paper is organized as follows. After giving a short review of the related literature in Section 2, we present the model and list examples of inequality indices in Section 3. Section 4 states the axioms and the main characterization theorem, the proof of which appears in Section 5. Section 6 concludes.

2 Related Literature

The axiomatic literature on inequality indices is quite vast. Weymark [29] defines a family of generalized Gini absolute inequality indices, and characterizes it within the class of societies with a fixed population. He also defines a family of generalized Gini relative inequality indices, which was axiomatically characterized by Ben Porath and Gilboa [5] within the class of societies with a fixed population and a fixed income. Yaari [30], Bossert [6] and Aaberge [1] provide characterizations of this family within a larger class of societies. Further characterizations of the Gini indices can be found in Thon [28], Donaldson and Weymark [13, 14], Yitzhaki [31], and Barret and Salles [4]. The family of generalized entropy measures has been studied by Cowell [10], Cowell and Kuga [11, 12], Shorrocks [23, 24], and Russell [22], to name a few. Finally, Atkinson [2] introduces and characterizes the family of Atkinson measures, which is further characterized by Lasso and Urrutia [19].[2]

One member of the family of general entropy indices is the Theil index, which has been introduced by Theil [26]. Theil [26, 27] shows this index to have the following useful property which, following Foster [15], we call Theil-Decomposability. The property can be described as follows. Partition a society into two groups of income earners. We can define its *within-group* inequality as the weighted average of the income inequality levels of the two groups, the weights being the income shares of each group. We can also define the *between-group* inequality as the inequality level of the original society after smoothing the income of each group. In other words, between-group inequality is the inequality that would result if there was no within-group inequality. It turns out that no matter how the original society is partitioned, Theil's index measures its income inequality as the sum of the within-group and between-group inequalities.

Bourguignon [7] used this decomposability property to axiomatically characterize the Theil index. In particular, he showed that it is the only twice differentiable index that satisfies various uncontroversial axioms as well as Theil-Decomposability. Foster [15] shows that the requirement of twice differentiability can be replaced by continuity.[3] In this paper we show on a class of continuous distributions that we can replace the cardinal axiom of Theil-Decomposability by weaker ordinal axioms and still obtain the Theil inequality ordering.

Our proof is very different from those of Bourguignon [7] and Foster [15]. Bourguignon heavily relies on the twice differentiability of the index. Foster, in turn, relies on Lee's [20] theorem to show that a particular restriction of any index that satisfies Theil decomposability (and other uncontroversial axioms) must be a multiple of Shannon's measure of entropy.[4] In contrast, we rely on a well-known characterization of the logarithmic functions to show

[2]For comprehensive surveys on income inequality measures, see Cowell [9] and Chakravarty [8].

[3]Foster's [15] main result further shows that continuity can be dispensed with by strengthening the Directness axiom and assuming the Pigou-Dalton principle of transfers instead.

[4]This restriction is the index applied only to two-person societies, with one person earning a proportion t of the total income and the other one the remaining $1 - t$.

that a specific index that satisfies our axioms, restricted to particularly simple societies, is in fact a logarithmic function.[5] While Foster's proof consists mainly of showing that a Theil-Decomposable index must be a multiple of Theil's measure, the most burdensome part of our proof consists in showing that an inequality ordering that satisfies our axioms can be represented by a Theil-Decomposable index. Once this is done, showing that this index is in fact Theil's measure is less difficult.[6] Our result is reminiscent of Frankel and Volij's [16] characterization of the Mutual Information measure of segregation. The main difference is that Frankel and Volij, owing to an extensive use of an axiom of symmetry among different ethnic groups and one of invariance to splitting of groups, are able to prove their result without resorting to decomposability. In our case, however, since there are no multiplicity of groups, we have to add a decomposability axiom.

An alternative but indirect proof of our result can be obtained if we strengthen our directness requirement to the Pigou-Dalton principle of transfers, and if we restrict attention to the class of discrete distributions. Indeed, Shorrocks [24, 25] shows that in this class, the generalized entropy indices are the only ones (up to a monotonic transformation) that satisfy symmetry, the Pigou-Dalton principle of transfers, replication invariance, homogeneity, continuity, and subgroup consistency. Therefore, by checking that none of the generalized entropy indices, except for the Theil index, satisfies our ordinal decomposability axiom, one obtains a full characterization of it. Shorrocks's proof, however, is restricted to the class of discrete distributions and makes heavy use of results on functional equations. Using the flexibility awarded by a larger class of distributions, we offer a direct and more elementary proof which we believe is easy to follow.

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[5]These simple societies are ones where a proportion $1 - t$ of the population has no income at all and the remaining proportion shares all of society's income evenly.

[6]We should point out that for this part of our proof we cannot rely on Foster's result because we deal with a class of societies that is larger than Foster's. Furthermore, his continuity axiom is different from ours.

3 Definitions

A *generalized society* is a non-decreasing, right-continuous real function $F : [0, \infty) \to \mathbb{R}_+$. For each $y \geq 0$, $F(y)$ represents the population mass with income less or equal y. We will restrict attention to generalized societies where income is bounded. Namely, for each society F, there is $\overline{y}$, such that $F(y) = F(\overline{y})$ for all $y \geq \overline{y}$.

For any generalized society F, we denote by $|F|$, the total level of income in F, and by $n(F)$ its total population. That is,

$$|F| = \int_0^\infty y \, dF \quad \text{and} \quad n(F) = \lim_{y \to \infty} F(y).$$

A *proper society* is a generalized society whose total level of income is positive, i.e $|F| > 0$. Note that if F and G are generalized societies, one of which is proper, then $F + G$ is also proper. Unless stated otherwise, whenever we refer to societies we mean proper ones. We denote by $\mathcal{F}$ the class of proper societies with bounded income. We denote by $\mathcal{F}_+$ the subclass of proper societies F such that $F(0) = 0$. Namely, with no agents with 0 income. Some indices are not well-defined for societies outside $\mathcal{F}_+$. For our proof to be valid it is important that the class is $\mathcal{F}$ and not just $\mathcal{F}_+$. The reason is that in order to show that any index that satisfies our axioms is the Theil index, we first restrict attention to a class of societies in which a positive proportion of the populations has 0 income. Only after showing that on this small class the order must be represented by the Theil index, can we extend the result to the whole class of societies. See Proposition 2 for details.

For each subset $E \subset \mathbb{R}$, $\mathbf{1}_E$ denotes its characteristic function. For any proper society $F \in \mathcal{F}$, we denote by $\overline{F}$ the smoothed society $n(F) \cdot \mathbf{1}_{[|F|/n(F), \infty)}$ that is obtained from F by redistributing F's income equally among its members. Also, for any $\alpha \geq 0$, and for any generalized society F, αF denotes the generalized society that is obtained from F by multiplying, for each $y \geq 0$, the mass of people with income less or equal y, by α. That is, αF is defined by $(\alpha F)(y) = \alpha F(y)$. Similarly, $F_{(\alpha)}$ denotes the generalized society that is obtained from F by multiplying, each individual's income by α. Formally, $F_{(\alpha)}$ is the function defined by $F_{(\alpha)}(y) = F(\alpha y)$.

An *inequality ordering* is a complete and transitive binary relation $\succcurlyeq$ on $\mathcal{F}$.[7] For any two societies F and G, $F \succcurlyeq G$ means that F's income distribution is at least as unequal as G's. Some orderings can be represented by an *inequality index*. An inequality index is a function $I : \mathcal{F} \to \mathbb{R}$ that assigns to each society in $\mathcal{F}$ a real number, that stands for the society's inequality level. We say that an inequality index $I : \mathcal{F} \to \mathbb{R}$ *represents* the inequality ordering $\succcurlyeq$ if for all societies $F, G \in \mathcal{F}$, $F \succcurlyeq G$ if and only if $I(F) \geq I(G)$.

3.1 Examples of inequality indices

Example 1 The *Theil index*, $T : \mathcal{F} \longrightarrow [0, \infty)$, is defined as follows.[8]
For all $F \in \mathcal{F}$,

$$T(F) = \int_0^\infty \frac{y}{|F|} \ln(\frac{n(F)}{|F|} y) dF(y).$$

The *Theil ordering* is the ordering represented by the Theil index.

Example 2 The *Second Theil index*, $T_0 : \mathcal{F}_+ \longrightarrow [0, \infty)$, is defined as follows.
For all $F \in \mathcal{F}_+$,

$$T_0(F) = \int_0^\infty \ln\left(\frac{|F|}{n(F)y}\right) \frac{1}{n(F)} dF(y).$$

Both T and T_0 belong to the family of generalized entropy indices. The remaining indices of this family are defined next.

Example 3 The *Generalized Entropy index*, $GE_\epsilon : \mathcal{F}_+ \longrightarrow [0, \infty)$, is defined as follows.
For all $F \in \mathcal{F}_+$,

$$GE_\epsilon(F) = \int_0^\infty \frac{\left(\frac{n(F)y}{|F|}\right)^\epsilon - 1}{\epsilon^2 - \epsilon} \frac{1}{n(F)} dF(y) \text{for } \epsilon \neq 0, 1.$$

[7] We denote by $\succ$ and $\sim$ the asymmetric and symmetric parts of $\succcurlyeq$.

[8] We adopt the convention that $0 \ln(0) = 0$.

Researchers are sometimes interested in decomposable inequality indices. Decomposable indices allow us to attribute total inequality to different factors. In particular, decomposable indices allow us to decompose total inequality into inequality *between* subsocieties and inequality *within* subsocieties. Bourguignon [7] and Foster [15] used the following version of decomposability in their characterizations of the Theil index.

Definition 1 [TD] We say that inequality index I is *Theil-decomposable* if for any two societies F and G,

$$I(F+G) = \frac{|F|}{|F+G|} I(F) + \frac{|G|}{|F+G|} I(G) + I(\overline{F} + \overline{G}). \tag{1}$$

The first two terms of the right hand side of (1) represent the inequality *within* F and G. This inequality is the income-weighted average of the inequality of the two subsocieties as measured by I. The last term of (1) represents the inequality *between* F and G, and is the inequality that would result if there was no inequality in either subsociety.

Note that Theil decomposability is a cardinal axiom. Nevertheless, it has very strong ordinal implications. In this paper we identify one of these ordinal implications and, together with other ordinal axioms, use them to characterize the Theil inequality ordering.

4 Axioms and the main result

The first two axioms embody the idea that we are interested in *relative* measures of income inequality.

Definition 2 [RI] We say that $\succcurlyeq$ satisfies replication invariance if for all $\alpha > 0$, and for all societies F, we have $\alpha F \sim F$.

Definition 3 [HOM] We say that $\succcurlyeq$ satisfies homogeneity if for all $\alpha > 0$, and for all societies F, we have $F_{(\alpha)} \sim F$.

Homogeneity states that only the relative distribution of income determines inequality. In other words, one does not need to know the units in which income is measured (dollars, euros, etc.) in each society to determine whether one society has a more or less equal distribution than another.[9] Replication invariance states that if we replicate a society by multiplying each individual by a fixed positive constant, then inequality remains unaffected. It is not the absolute number of people who have any given income level that matters, but their proportion in the population. It is easy to check that all the orderings listed in the previous section satisfy homogeneity and replication invariance.

The previous two axioms dictate that a particular change in the society does not affect its income inequality. The next axiom, on the other hand, dictates that other changes do have a certain effect. In fact, it is the only axiom that provides circumstances under which one society is more unequal than another.

Definition 4 [SD] We say that $\succcurlyeq$ satisfies strong directedness if for all two-income-group societies $F = n_1 \cdot \mathbf{1}_{[y_1,\infty)} + n_2 \cdot \mathbf{1}_{[y_2,\infty)}$, where $y_1 < y_2$, and $n_1, n_2 > 0$,

$$n_1 \cdot \mathbf{1}_{[y_1,\infty)} + n_2 \cdot \mathbf{1}_{[y_2,\infty)} \succ n(F) \cdot \mathbf{1}_{[|F|/n(F),\infty)}$$

Strong directedness is a stronger version of Foster [15]'s directedness. According to this axiom, if one divides an egalitarian society into two income groups by transferring income from some individuals to others, one obtains a new society with a more unequal distribution of income. It is easy to check that all the indices listed in the previous section represent orderings that satisfy strong directedness.

The next axiom is an ordinal implication of Theil-Decomposability.

Definition 5 [IND] We say that $\succcurlyeq$ satisfies independence, if for all two societies F_1 and F_2

[9] A related property is the unit-consistency axiom introduced by Zheng [32]. It guarantees that, as long as income is measured in the same unit in all societies, inequality rankings are independent of this unit. We don't know the extent to which substituting this weaker axiom for Homogeneity would affect our results. Examples of unit-consistent measures may be found in Zheng [32, 33] and del Rio and Alonso-Villar [21].

such that $n(F_1) = n(F_2)$ and $|F_1| = |F_2|$, and for all generalized societies F,

$$F_1 \succcurlyeq F_2 \Leftrightarrow F_1 + F \succcurlyeq F_2 + F.$$

Independence is essentially what is known as *subgroup consistency*, which is closely related to the notion of *aggregativity* of an index (see Shorrocks [24, 25]). It says that if a given society is composed of two regions, and one of its regions' income becomes more unequally distributed, then the income distribution of the whole society becomes more unequal as well. The satisfaction of this axiom justifies the application of distributive policies in subregions in order to obtain results in the whole region. To illustrate, in order to reduce income inequality in Asia, one would want to apply a policy that reduces inequality in India. But this would be justified only if our measure of inequality satisfies IND. Otherwise, it may well be the case that by reducing inequality in India we end up increasing inequality in Asia. An immediate consequence of Shorrocks [24, 25] results is that many indices, including the Gini index, fail to satisfy Independence.

As mentioned above, IND is an ordinal implication of Theil-Decomposability. To see this, let $F_1, F_2 \in \mathcal{F}$ be two societies such that $|F_1| = |F_2|$ and $n(F_1) = n(F_2)$, and let $F \in \mathcal{F}$ be another society. Also, let $\alpha = \frac{|F_1|}{|F_1+F|} = \frac{|F_2|}{|F_2+F|} > 0$. Then, if I is a Theil-Decomposable index,

$$\begin{aligned} I(F_1+F) - I(F_2+F) &= \alpha I(F_1) + I(\overline{F_1} + \overline{F}) - \alpha I(F_2) - I(\overline{F_2} + \overline{F}) \\ &= \alpha(I(F_1) - I(F_2)), \end{aligned}$$

since $\overline{F_1} = \overline{F_2}$. Hence, I represents an ordering that satisfies IND.

The next axiom is another ordinal implication of Theil-decomposability.

Definition 6 [DEC] We say that $\succcurlyeq$ satisfies ordinal decomposability if for all two societies F_1, F_2, such that $|F_1| = |F_2|$, and for any generalized society F,

$$F_1 + \overline{F} \succcurlyeq F_2 + \overline{F} \Rightarrow F_1 + F \succcurlyeq F_2 + F. \tag{2}$$

Ordinal decomposability states the following. Suppose we want to compare two societies, F_1+F and F_2+F, in terms of income distribution. These two societies may not have the same population but they do have the same total income. Further, suppose that these two societies share a common subsociety, F. That is, their intersection is not empty. To illustrate, think of Russia, which belongs both to Europe and to Asia, and assume that Europe and Asia have the same total income. Ordinal decomposability dictates that whether or not one society is more unequal than the other is independent of the income distribution in the common subsociety. Continuing with our example, DEC states that whether or not Europe has a more unequal distribution than Asia is independent of how income is distributed within Russia. In particular, it is enough to know whether Europe would have a more unequal distribution than Asia if Russia's income was equally distributed among Russia's population.

This axiom suggests that in some circumstances one could identify and isolate the inequality within a subsociety from the inequality of the whole society. To see this, note that for any F, the difference between society $(F_1 + \overline{F})$ and society $(F_1 + F)$ is that in the first society the inequality *within* F has been eliminated while in the second it has not. Therefore the requirement (2) in DEC suggests that income inequality in $(F_1 + F)$ consists of the inequality in $(F_1 + \overline{F})$ and of a term that depends only on the inequality within F and on $|F_1|$. This kind of *income dependent* decomposability is much weaker than Theil Decomposability, but turns out to be sufficient, together with IND and the other axioms, to imply it.

As mentioned above, DEC is weaker than Theil-Decomposability. To see this, let $F_1, F_2, F \in \mathcal{F}$, such that $|F_1| = |F_2|$, and let $\alpha = \frac{|F_1|}{|F_1+F|} = \frac{|F_2|}{|F_2+F|}$. Assume that the index I satisfies TD. Then,

$$
\begin{aligned}
(F_1 + \overline{F}) &\succcurlyeq (F_2 + \overline{F}) && \Leftrightarrow \\
\alpha I(F_1) + (1-\alpha) I(\overline{F}) + I(\overline{F_1} + \overline{F}) &\geq \alpha I(F_2) + (1-\alpha) I(\overline{F}) + I(\overline{F_2} + \overline{F}) && \Leftrightarrow \\
\alpha I(F_1) + (1-\alpha) I(F) + I(\overline{F_1} + \overline{F}) &\geq \alpha I(F_2) + (1-\alpha) I(F) + I(\overline{F_2} + \overline{F}) && \Leftrightarrow \\
(F_1 + F) &\succcurlyeq (F_2 + F)
\end{aligned}
$$

which means that DEC is satisfied.

The reader may wonder what the implications are of replacing the condition $|F_1| = |F_2|$ in the definition of DEC with the dual condition $n(F_1) = n(F_2)$. In that case we would obtain a *population dependent* decomposability axiom. As it turns out, among the generalized entropy indices, only the second Theil measure, T_0, satisfies this new axiom. Therefore, by restricting attention to the class of finite population societies, this axiom along with the other axioms used in Shorrocks [25] fully characterize T_0. A modification of our proof, however, would not suffice to obtain a characterization of this index in the larger class that we consider in this paper. The reason is that our proof, in particular Proposition 2, crucially exploits the fact that there are societies with zero per-capita income groups, and the second Theil index is not defined for such societies.

The last axiom is a technical but standard continuity requirement that states that "similar" societies have "similar" levels of income inequality. For any function $F \in \mathcal{F}$, its L_1-norm is defined by $\|F\| = \int_0^\infty |F(y)|\, dy$. We say that the sequence $\{F_k\}$ converges to F, denoted $F_k \to F$, if $\|F_k - F\| \to 0$.

Definition 7 [C] The inequality ordering $\succcurlyeq$ satisfies continuity (in the L_1-norm) if for any sequence of pairs of societies, $\{F_k, G_k\}$ with $F_k \succcurlyeq G_k$ for all k and $F_k \to F$ and $G_k \to G$, we have $F \succcurlyeq G$.

Needless to say, this axiom is weaker than directly assuming that $\succcurlyeq$ is represented by a continuous index.

We are now ready to state our main result.

Theorem 1 *An inequality ordering defined on $\mathcal{F}$ satisfies homogeneity, replication invariance, independence, ordinal decomposability, strong directedness, and continuity if and only if it can be represented by the Theil inequality index.*

5 Proof of Theorem 1

It is easy to check that the Theil ordering satisfies HOM, RI, SD, and C. It is well known that the Theil index satisfies Theil-Decomposability. Therefore it satisfies IND and DEC as well.

Now, let $\succcurlyeq$ be an inequality ordering on $\mathcal{F}$ that satisfies homogeneity, replication invariance, independence, ordinal decomposability, strong directedness, and continuity. We will show that is represented by the Theil index.

We say that general society S is a *simple society* if it is of the form

$$S = \sum_{k=1}^{K} n_k \cdot \mathbf{1}_{[y_k,\infty)}$$

where $0 \leq y_1 < \cdots < y_K$. Note that a simple society is a proper society if $n_k y_k > 0$ for some k. Denote by $\mathcal{F}_S \subset \mathcal{F}$ the class of simple, proper societies. We shall first show that the only ordering that satisfies the axioms on $\mathcal{F}_S$ is the Theil ordering. We will later extend this result to the whole class $\mathcal{F}$ of proper societies.

Let $S_0 = \mathbf{1}_{[1,\infty)}$ be the society with population mass 1 and a uniformly distributed income of one, and let $S_{1/2} = 1/2 \cdot \mathbf{1}_{[0,\infty)} + 1/2 \cdot \mathbf{1}_{[2,\infty)}$ be the society with population mass 1, in which half of the population has income 0, and the other half has income 2. Note that $S_{1/2}$ has income 1. Also note that by SD, $S_{1/2} \succ S_0$.

Lemma 1 *All societies where total income is equally distributed among the population have the same degree of income inequality. Further, for all societies $S \in \mathcal{F}_S$, $S \succcurlyeq S_0$.*

Proof. Let $F = n \cdot \mathbf{1}_{[y,\infty)}$ be a society with equally distributed income. By HOM, RI, $F \sim \frac{1}{n} F_{(1/y)} = \mathbf{1}_{[1,\infty)} = S_0$.

Now let $S = \sum_{k=1}^{K} n_k \cdot \mathbf{1}_{[y_k,\infty)}$. We will use induction to show that $S \succcurlyeq S_0$. If $K = 1$, then, by the previous step $S \sim S_0$. Assume that result holds for $K = m \geq 1$, and let now $K = m + 1$. By SD,

$$n_{K-1} \cdot \mathbf{1}_{[y_{K-1},\infty)} + n_K \cdot \mathbf{1}_{[y_K,\infty)} \succcurlyeq (n_{K-1} + n_K) \cdot \mathbf{1}_{[y,\infty)}$$

where y is the level of income that satisfies $(n_{K-1}+n_K)y = n_{K-1}y_{K-1}+n_K y_K$. If $m=2$, by the previous step we are done. Otherwise, if $m>2$, by IND,

$$\sum_{k=1}^{K} n_k \cdot \mathbf{1}_{[y_k,\infty)} \succcurlyeq \sum_{k=1}^{K-2} n_k \cdot \mathbf{1}_{[y_k,\infty)} + (n_{K-1}+n_K)\cdot \mathbf{1}_{[y,\infty)}$$

By the induction hypothesis,

$$\sum_{k=1}^{K-2} n_k \cdot \mathbf{1}_{[y_k,\infty)} + (n_{K-1}+n_K)\cdot \mathbf{1}_{[y,\infty)} \succcurlyeq S_0.$$

Q.E.D.

Lemma 2 *Let $S' \in \mathcal{F}_S$ be a society with population 1 and income 1 such that $S' \succ S_0$. If $0 \leq \alpha < \beta < 1$, then $\beta S' + (1-\beta)S_0 \succ \alpha S' + (1-\alpha)S_0$.*

Proof. By RI, $(\beta-\alpha)S' \succ (\beta-\alpha)S_0$. By IND,

$$\alpha S' + (\beta-\alpha)S' + (1-\beta)S_0 \succ \alpha S' + (\beta-\alpha)S_0 + (1-\beta)S_0,$$

which can be written as, $\beta S' + (1-\beta)S_0 \succ \alpha S' + (1-\alpha)S_0$. Q.E.D.

Lemma 3 *Let $S' \in \mathcal{F}_S$ be a society with population 1 and income 1 such that $S' \succ S_0$. Then, for any society $S \in \mathcal{F}_S$ such that $S' \succcurlyeq S \succcurlyeq S_0$, there is a unique $\alpha^* \in [0,1]$ such that*

$$S \sim \alpha^* S' + (1-\alpha^*)S_0$$

Proof. By C, the sets $\{G : G \succcurlyeq S\}$ and $\{G : S \succcurlyeq G\}$ are closed in $\mathcal{F}$. Consequently, the sets $\{\alpha \in [0,1] : \alpha S' + (1-\alpha)S_0 \succcurlyeq S\}$ and $\{\alpha \in [0,1] : S \succcurlyeq \alpha S' + (1-\alpha)S_0\}$ are closed in $\mathbb{R}$. Since $S' \succcurlyeq S \succcurlyeq S_0$, they are not empty. Since $\succcurlyeq$ is complete, their union is $[0,1]$. Therefore, since the unit interval is connected, the intersection of the two sets is not empty. By Lemma 2, this intersection must contain a single element. This single element is the α^* we are looking for. Q.E.D.

Lemma 4 *For any society $S \in \mathcal{F}_S$ such that $|S| = n(S) = 1$, there is a unique $\alpha^* \geq 0$ such that $S + \alpha^* S_0 \sim S_0 + \alpha^* S_{1/2}$.*

Proof. If $S_{1/2} \succcurlyeq S \succcurlyeq S_0$, then by Lemma 3, and since $S_{1/2} \succ S_0$, there is a unique $\alpha^* \in [0,1]$ such that $S \sim \alpha^* S_{1/2} + (1-\alpha^*)S_0$. Then, by IND,

$$\begin{aligned} S + \alpha^* S_0 &\sim \alpha^* S_{1/2} + (1-\alpha^*)S_0 + \alpha^* S_0 \\ &= \alpha^* S_{1/2} + S_0. \end{aligned}$$

If, on the other hand, $S \succ S_{1/2}$, by Lemma 3 there is a unique $\beta^* \in [0,1]$ such that $\beta^* S + (1-\beta^*)S_0 \sim S_{1/2}$. Since $S_{1/2} \succ S_0$, $\beta^* > 0$. Then,

$$\begin{aligned} S + \tfrac{(1-\beta^*)}{\beta^*} S_0 &\sim \tfrac{1}{\beta^*} S_{1/2} && \text{by RI} \\ S + \tfrac{(1-\beta^*)}{\beta^*} S_0 + S_0 &\sim \tfrac{1}{\beta^*} S_{1/2} + S_0 && \text{by IND} \\ S + \tfrac{1}{\beta^*} S_0 &\sim \tfrac{1}{\beta^*} S_{1/2} + S_0 \end{aligned}$$

Therefore, $\frac{1}{\beta^*}$ is the α^* we are looking for. Q.E.D.

Lemma 4 allows us to define an index $r : \mathcal{F}_S \to \mathbb{R}$ by

$$r(S) = \alpha,$$

where α is the unique number that satisfies $\widehat{S} + \alpha S_0 \sim S_0 + \alpha S_{1/2}$, and $\widehat{S}$ is the society that is obtained from S by normalizing its population and income to 1.

Lemma 5 *The index r represents the inequality order $\succcurlyeq$ on $\mathcal{F}_S$.*

Proof. Let S and S' be two societies in $\mathcal{F}_S$ and assume that $S' \succcurlyeq S$. By RI and HOM we can assume that $|S| = |S'| = 1$ and $n(S) = n(S') = 1$. Let α and α' be defined by

$$S + \alpha S_0 \sim S_0 + \alpha S_{1/2} \tag{3}$$

$$S' + \alpha' S_0 \sim S_0 + \alpha' S_{1/2}. \tag{4}$$

We need to show that $\alpha' \geq \alpha$. Assume by contradiction that $\alpha > \alpha'$. Then,

$$\begin{aligned} S' + \alpha S_0 &\succcurlyeq S + \alpha S_0 && \text{by IND} \\ &\sim S_0 + \alpha S_{1/2} && \text{by (3)} \\ &= S_0 + \alpha' S_{1/2} + (\alpha - \alpha') S_{1/2} \\ &\succ S_0 + \alpha' S_{1/2} + (\alpha - \alpha') S_0 && \text{by IND and } S_{1/2} \succ S_0, \\ &\sim S' + \alpha' S_0 + (\alpha - \alpha') S_0 && \text{by (4) and IND} \\ &= S' + \alpha S_0 \end{aligned}$$

which cannot be true. Q.E.D.

The next proposition implies that the index r satisfies Theil-Decomposability.

Proposition 1 *Let S and S' be two simple societies such that $S \in \mathcal{F}_S$. Then*

$$r(S+S') = \frac{|S|}{|S+S'|} r(S) + r\left(\overline{S}+S'\right)$$

Proof. Let S and S' be two simple societies with populations n and m, respectively. By RI and HOM, we can assume without loss of generality that $n+m=1$, and $|S+S'|=1$. Let $\gamma = r(\overline{S}+S')$ and $\alpha = r(S)$. Then, by definition of r,

$$\left(\overline{S}+S'\right) + \gamma S_0 \sim S_0 + \gamma S_{1/2} \tag{5}$$

$$\widehat{S} + \alpha S_0 \sim S_0 + \alpha S_{1/2}. \tag{6}$$

where $\widehat{S}$ is the society that is obtained from S by normalizing its population and income so that they are both equal to 1. We need to show that

$$(S+S') + (|S|\,\alpha + \gamma) S_0 \sim S_0 + (|S|\,\alpha+\gamma) S_{1/2}. \tag{7}$$

Choose $k \in \mathbb{N}$ such that $k > 1 + \alpha\,|S|$. Denote $S^*_{1/2} = \frac{n}{2} \cdot \mathbf{1}_{[0,\infty)} + \frac{n}{2} \cdot \mathbf{1}_{[\frac{2|S|}{n},\infty)}$. This society has population n and income $|S|$. It follows from (6), using RI and HOM, that

$$S + \alpha\overline{S} \sim \overline{S} + \alpha S^*_{1/2}. \tag{8}$$

Adding $(k-1)(S_0 + \gamma S_{1/2})$ to both sides of Equation (5), we obtain

$$\begin{aligned}
\overline{S} + \overbrace{S' + \gamma S_0 + (k-1)(S_0+\gamma S_{1/2})}^{Z_1} \quad &\sim \quad k(S_0+\gamma S_{1/2}) && \text{by IND} \\
&\sim \quad \tfrac{k}{|S|}(\overline{S} + \gamma S^*_{1/2}) && \text{by HOM and RI} \\
&= \quad \overline{S} + \overbrace{\frac{k}{|S|}((\underbrace{1-\frac{|S|}{k}}_{>0})\,\overline{S} + \gamma S^*_{1/2})}^{Z_2}
\end{aligned}$$

Note that since $|S| < 1$ and $\alpha \geq 0$, our choice of k implies that $|S|/k < 1$, and therefore subsociety Z_2 is well-defined. Since $|Z_1| = |Z_2| = |S'| + k\gamma + (k-1)$, by DEC,

$$S + \overbrace{S' + \gamma S_0 + (k-1)(S_0 + \gamma S_{1/2})}^{Z_1} \sim S + \overbrace{\frac{k}{|S|}((1 - \frac{|S|}{k})\overline{S} + \gamma S^*_{1/2})}^{Z_2}.$$

Rewriting the right hand side term,

$$\begin{aligned} S + S' + \gamma S_0 + (k-1)(S_0 + \gamma S_{1/2}) &\sim S + \alpha\overline{S} + \tfrac{k}{|S|}((\underbrace{1 - \frac{(1+\alpha)|S|}{k}}_{>0})\,\overline{S} + \gamma S^*_{1/2}) \\ &\sim \overline{S} + \alpha S^*_{1/2} + \tfrac{k}{|S|}((1 - \tfrac{(1+\alpha)|S|}{k})\overline{S} + \gamma S^*_{1/2}) \\ &\sim |S|\,(S_0 + \alpha S_{1/2}) + (\underbrace{k - 1 - \alpha\,|S|}_{>0})S_0 + k\gamma S_{1/2} \\ &= S_0 + (\alpha\,|S| + \gamma)\,S_{1/2} + (k - (1+\alpha)|S|)S_0 + (k-1)\,\gamma S_{1/2} \end{aligned}$$

where the second line follows from (8) and IND, and the third line from HOM and RI. On the other hand,

$$S + S' + \gamma S_0 + (k-1)(S_0 + \gamma S_{1/2}) = S + S' + (\alpha\,|S| + \gamma)S_0 + (k - 1 - \alpha\,|S|)\,S_0 + (k-1)\gamma S_{1/2}.$$

As a result, denoting $k^* = (k - 1 - \alpha\,|S|)$, we obtain

$$S + S' + (\alpha\,|S| + \gamma)S_0 + \left(k^* S_0 + (k-1)\gamma S_{1/2}\right) \sim S_0 + (\alpha\,|S| + \gamma)\,S_{1/2} + \left(k^* S_0 + (k-1)\gamma S_{1/2}\right)$$

Since $S + S' + (\alpha\,|S| + \gamma)S_0$ and $S_0 + (\alpha\,|S| + \gamma)\,S_{1/2}$ have the same population and income, we can apply IND and obtain

$$S + S' + (\alpha\,|S| + \gamma)S_0 \sim S_0 + (\alpha\,|S| + \gamma)\,S_{1/2},$$

which is what we wanted to prove. Q.E.D.

Corollary 1 *Let $S_1, \ldots, S_K$ be K societies in $\mathcal{F}_S$. Also let $S = \sum_{k=1}^{K} S_k$. Then*

$$r\,(S) = \sum_{k=1}^{K} \frac{|S_k|}{|S|} r\,(S_k) + r\left(\sum_{k=1}^{K} \overline{S_k}\right).$$

Proof. The proof is by induction and is left to the reader. Q.E.D.

We now define a subclass of simple societies. For each $\alpha \in (0,1)$, let $S_\alpha = \alpha \cdot \mathbf{1}_{[0,\infty)} + (1-\alpha) \cdot \mathbf{1}_{[1/(1-\alpha),\infty)}$ be the society with population mass 1 in which a proportion α of the population has income 0, and the proportion $(1-\alpha)$ of the population has income $1/(1-\alpha)$. The next proposition shows that r, when applied to these societies, induces a well-known function.

Proposition 2 *For all* $\alpha \in (0,1]$, $r(S_{1-\alpha}) = -\log_2 \alpha$.

Proof. Let $h : (0,1] \to \mathbb{R}$ be defined by $h(\alpha) = r(S_{1-\alpha})$. By definition of r,

$$h(\alpha) \geq 0 \quad \text{for all } \alpha \in (0,1]. \tag{9}$$

Also,

$$h(1/2) = r(S_{1/2}) = 1. \tag{10}$$

We will now show that

$$h(pq) = h(p) + h(q) \quad \text{for all } p, q \in (0,1]. \tag{11}$$

To see this, note that

$$\begin{aligned} S_{1-pq} &= (1-pq) \cdot \mathbf{1}_{[0,\infty)} + pq \cdot \mathbf{1}_{[\frac{1}{pq},\infty)} \\ &= (1-q) \cdot \mathbf{1}_{[0,\infty)} + q\,(1-p) \cdot \mathbf{1}_{[0,\infty)} + pq \cdot \mathbf{1}_{[\frac{1}{pq},\infty)} \\ &= \left(q\,(1-p) \cdot \mathbf{1}_{[0,\infty)} + pq \cdot \mathbf{1}_{[1/(pq),\infty)}\right) + (1-q) \cdot \mathbf{1}_{[0,\infty)}. \end{aligned}$$

Therefore, by Proposition 1, and using HOM and RI,

$$\begin{aligned} r(S_{1-pq}) &= r\left(q\,(1-p) \cdot \mathbf{1}_{[0,\infty)} + pq \cdot \mathbf{1}_{[1/(pq),\infty)}\right) + r\left(q \cdot \mathbf{1}_{[1/q,\infty)} + (1-q) \cdot \mathbf{1}_{[0,\infty)}\right) \\ &= r\left((1-p) \cdot \mathbf{1}_{[0,\infty)} + p \cdot \mathbf{1}_{[1/p,\infty)}\right) + r\left(q \cdot \mathbf{1}_{[1/q,\infty)} + (1-q) \cdot \mathbf{1}_{[0,\infty)}\right) \\ &= r\left(S_{1-p}\right) + r(S_{1-q}), \end{aligned}$$

which shows that (11) holds. It is known that the only function on $(0,1]$ that satisfies (9-11) is $-\log_2$.[10] Q.E.D.

[10] See Theorem 0.2.5 in Aczél and Daróczy [3].

Proposition 3 *The index r is a positive multiple of the Theil index.*

Proof. Let $S = \sum_{k=1}^{K} n_k \cdot \mathbf{1}_{[y_k,\infty)} \in \mathcal{F}_S$ be a society where $0 \leq y_1 < \cdots < y_K$. Assume without loss of generality that $n_k > 0$ for $k = 1, \ldots, K$. We need to show that $r(S) = aT(S)$ for some $a > 0$. If $K = 1$, the result is obvious. So assume $K \geq 2$. By RI we can assume without loss of generality that $n(S) = 1$. Similarly, by HOM we can assume without loss of generality that $\sum_{k=1}^{K} y_k = 1$. Therefore $|S|^2 < |S| = \sum_{k=1}^{K} n_k y_k < 1$. Also, $y_k|S| < 1$ for $k = 1, \ldots K$. We need to show that $r(S)$ is a multiple of $T(S)$, which in the case of our simple society, can be written as

$$T(S) = \sum_{k=1}^{K} \frac{n_k y_k}{|S|} \left(\ln \frac{y_k}{|S|} \right).$$

Assume first that $y_1 > 0$, and define

$$S^k = n_k(1 - y_k|S|) \cdot \mathbf{1}_{[0,\infty)} + n_k y_k |S| \cdot \mathbf{1}_{[\frac{1}{|S|},\infty)}.$$

Note that $\overline{S^k} = n_k \cdot \mathbf{1}_{[y_k,\infty)}$ and therefore $\sum_{k=1}^{K} \overline{S^k} = S$. Therefore, by Corollary 1,

$$r(\sum_{k=1}^{K} S^k) = \sum_{k=1}^{K} \frac{n_k y_k}{|S|} r\left(S^k\right) + r(S).$$

which can be written as

$$r(S) = r(\sum_{k=1}^{K} S^k) - \sum_{k=1}^{K} \frac{n_k y_k}{|S|} r\left(S^k\right).$$

Note that by RI, and HOM, for all k

$$\begin{aligned} S^k &\sim (1 - y_k|S|) \cdot \mathbf{1}_{[0,\infty)} + y_k|S| \cdot \mathbf{1}_{[\frac{1}{|S|},\infty)} \\ &\sim (1 - y_k|S|) \cdot \mathbf{1}_{[0,\infty)} + y_k|S| \cdot \mathbf{1}_{[\frac{1}{y_k|S|},\infty)} = S_{1-y_k|S|}. \end{aligned}$$

Also, by HOM,

$$\begin{aligned} \sum_{k=1}^{K} S^k &\sim \sum_{k=1}^{K} \left(n_k(1 - y_k|S|) \cdot \mathbf{1}_{[0,\infty)} + n_k y_k |S| \cdot \mathbf{1}_{[\frac{1}{|S|^2},\infty)} \right) \\ &= \left(\sum_{k=1}^{K} n_k - \sum_{k=1}^{K} n_k y_k |S| \right) \cdot \mathbf{1}_{[0,\infty)} + \sum_{k=1}^{K} n_k y_k |S| \cdot \mathbf{1}_{[\frac{1}{|S|^2},\infty)} \\ &= \left(1 - |S|^2\right) \cdot \mathbf{1}_{[0,\infty)} + |S|^2 \cdot \mathbf{1}_{[\frac{1}{|S|^2},\infty)} = S_{1-|S|^2}. \end{aligned}$$

Therefore

$$\begin{aligned}
r(S) &= r(S_{1-|S|^2}) - \sum_{k=1}^{K} \frac{n_k y_k}{|S|} r\left(S_{1-y_k|S|}\right) \\
&= \log_2 \frac{1}{|S|^2} + \sum_{k=1}^{K} \frac{n_k y_k}{|S|} \log_2 y_k |S| \\
&= \sum_{k=1}^{K} \frac{n_k y_k}{|S|} \left(\log_2 \frac{1}{|S|^2} + \log_2 y_k |S| \right) \\
&= \sum_{k=1}^{K} \frac{n_k y_k}{|S|} \left(\log_2 \frac{y_k}{|S|} \right) \\
&= \frac{T(S)}{\ln 2}.
\end{aligned}$$

Assume now that $y_1 = 0$, so that $S = n_1 \cdot \mathbf{1}_{[0,\infty)} + \sum_{k=2}^{K} n_k \cdot \mathbf{1}_{[y_k,\infty)}$. Namely, there is a positive mass of agents with 0 income. Denote $S' = \sum_{k=2}^{K} n_k \cdot \mathbf{1}_{[y_k,\infty)}$. Then, by Proposition 1,

$$r(S) = r(S') + r(\overline{S'} + n_1 \cdot \mathbf{1}_{[0,\infty)}).$$

But, using HOM,

$$\begin{aligned}
\overline{S'} + n_1 \cdot \mathbf{1}_{[0,\infty)} &= n_1 \cdot \mathbf{1}_{[0,\infty)} + (1 - n_1) \cdot \mathbf{1}_{[|S|/(1-n_1),\infty)} \\
&\sim n_1 \cdot \mathbf{1}_{[0,\infty)} + (1 - n_1) \cdot \mathbf{1}_{[1/(1-n_1),\infty)} \\
&= S_{n_1}
\end{aligned}$$

where the second line follows from HOM. Then, by Proposition 2,

$$\begin{aligned}
r(\overline{S'} + n_1 \cdot \mathbf{1}_{[0,\infty)}) &= r(S_{n_1}) \\
&= -\log_2(1 - n_1).
\end{aligned}$$

Consequently,

$$\begin{aligned}
r(S) &= r(S') - \log_2(1 - n_1) \\
&= \frac{T(S')}{\ln 2} - \log_2(1 - n_1) \\
&= \frac{T(S)}{\ln 2}.
\end{aligned}$$

We have shown that the only ordering on $\mathcal{F}_S$ that satisfies the axioms is the Theil ordering. We now extend this result to the whole class $\mathcal{F}$. Let $\succcurlyeq$ be the Theil ordering and let $\succcurlyeq^*$ be an order on $\mathcal{F}$ that satisfies the forgoing axioms. We must show that $\succcurlyeq^*$ and $\succcurlyeq$ are one and the same order. Let $F, G \in \mathcal{F}$ be two societies such that $F \succcurlyeq G$. We must show that $F \succcurlyeq^* G$. In order to do so, it is enough to find a sequence $\{F_n, G_n\}$ of pairs of simple societies such that for all n, $F_n, G_n \in \mathcal{F}_S$, $F_n \succcurlyeq F$ and $G \succcurlyeq G_n$, and such that $F_n \to F$ and $G_n \to G$. To see why, note that once such sequence is found, we will have, by transitivity of $\succcurlyeq$, that $F_n \succcurlyeq G_n$ for all n. Since $\succcurlyeq^*$ satisfies the axioms on $\mathcal{F}_S$, by Theorem 1 it must coincide with $\succcurlyeq$ there. Consequently, we will have found a sequence of pairs $\{F_n, G_n\}$ such that that $F_n \to F$ and $G_n \to G$, with $F_n \succcurlyeq^* G_n$ for all n. By L_1-continuity we will have $F \succcurlyeq^* G$, the desired result. We now find the required sequence. Let $\overline{y} \in \arg\max F(y) \cap \arg\max G(y)$ be a common upper bound to the individual income levels in societies F and G. For each $n \in \mathbb{N}$, partition the interval $[0, \overline{y})$ into intervals of equal length of the type $[y_{k-1}^n, y_k^n) := [(k-1)\overline{y}/n, k\overline{y}/n)$. Note that the length of the intervals is $\overline{y}/n$. In order to build F_n, for each $k = 1, \ldots n$, we will regressively redistribute the income of the agents in the income bracket $(y_{k-1}^n, y_k^n]$, in a way so that some agents there will be impoverished and end up with y_{k-1}^n, and the remaining agents will be enriched and end up with y_k^n. Formally, for $k = 1, \ldots n$, let p_k^n be implicitly defined by

$$p_k^n y_{k-1}^n + \left(F\left(y_k^n\right) - F\left(y_{k-1}^n\right) - p_k^n\right) y_k^n = \int_{y_{k-1}^n}^{y_k^n} y \, dF. \tag{12}$$

The value p_k^n represents the mass of agents that are impoverished, and $\left(F\left(y_k^n\right) - F\left(y_{k-1}^n\right) - p_k^n\right)$ is the mass of people that are enriched. Equation (12) ensures that the total income of the agents in the original income bracket $(y_{k-1}^n, y_k^n]$ remains the same. Define now,

$$F_n = \left(F\left(y_0^n\right) + p_1^n\right) \cdot \mathbf{1}_{[y_0^n, \infty)} + \sum_{k=1}^{n} \left(F\left(y_k^n\right) + p_{k+1}^n - F\left(y_{k-1}^n\right) - p_k^n\right) \cdot \mathbf{1}_{[y_k^n, \infty)}.$$

Figure 1 illustrates the construction of F_n. As explained above, F_n is the society that is obtained from F by, for $k = 1, \ldots n$, redistributing the income of the agents in the income bracket $(y_{k-1}^n, y_k^n]$ among them, assigning income y_{k-1}^n to p_k^n of them and income y_k^n to the remaining $\left(F\left(y_k^n\right) - F\left(y_{k-1}^n\right) - p_k^n\right)$. By construction, F second order stochastically dominates F_n and since the Theil index is consistent with second order stochastic ordering, we have that $F_n \succcurlyeq F$.

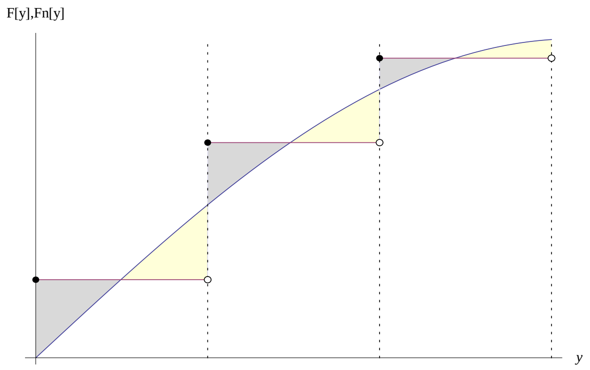

Figure 1: Societies F and F_n

The construction of the societies G_n is similar. In order to do so, we will progressively redistribute the income of the income bracket $[y^n_{k-1}, y^n_k)$ so that all agents there end up with the same income level. Formally, let

$$\overline{y}^n_k = \begin{cases} \frac{\int_{y^n_{k-1}}^{y^n_k} y \, dG}{G(y^n_k) - G(y^n_{k-1})} & \text{if } G(y^n_k) > G(y^n_{k-1}) \\ (k - 1/2)\overline{y}/n & \text{if } G(y^n_k) = G(y^n_{k-1}) \end{cases}$$

be the mean income of the agents in the income bracket $[y^n_{k-1}, y^n_k)$ (if there are no agents in this income bracket, the mean income is arbitrarily defined as the midpoint of the interval). Letting, $\overline{y}^n_{n+1} = \overline{y}$, Define

$$G_n = G(0) \cdot 1_{[0,\infty)} + \sum_{k=1}^{n} \big(G(y^n_k) - G(y^n_{k-1})\big) \cdot 1_{[\overline{y}^n_k,\infty)}.$$

Figure 2 illustrates the construction of G_n. The simple society G_n is obtained from G by redistributing equally the total income of each income bracket $[y^n_{k-1}, y^n_k)$. (If there are no agents in that bracket, G_n and G coincide there). By construction, G_n second-order stochastically dominates G. It is known that the Theil index is consistent with second-order stochastic dominance, therefore, $G \succcurlyeq G_n$.

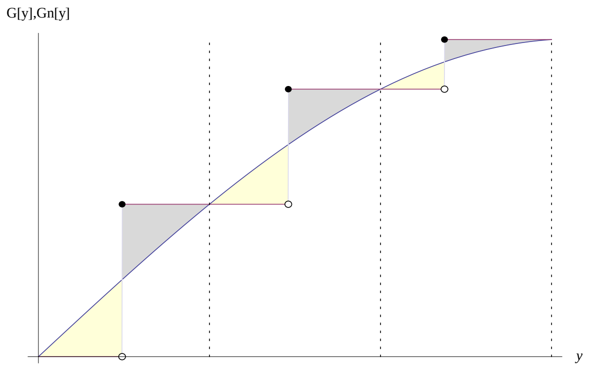

Figure 2: Societies G and G_n

It remains to show that $F_n \to F$ and $G_n \to G$. In order to show $F_n \to F$, by the bounded convergence theorem, it is enough to prove that F_n converges to F pointwise almost everywhere. Since F is non-decreasing, it is continuous almost everywhere. Therefore, it is enough to show that $F_n(y)$ converges to $F(y)$ for every point at which F is continuous. Let $y \in [0, \overline{y})$ one such point, and for any $n \in \mathbb{N}$, let $[y_{k-1}^n, y_k^n)$ be the interval that contains y. Note that since $p_k^n \leq F(y_k^n) - F(y_{k-1}^n)$, $F(y_{k-1}^n) \leq F_n(y) = F(y_{k-1}^n) + p_k^n \leq F(y_k^n)$. Since F is non-decreasing, we also have $F(y_{k-1}^n) \leq F(y) \leq F(y_k^n)$. Consequently, $|F_n(y) - F(y)| \leq \left|F(y_k^n) - F(y_{k-1}^n)\right|$. Now, let $\varepsilon > 0$. We need to show that there is n_0 such that for all $n > n_0$, $|F_n(y) - F(y)| \leq \varepsilon$. Since F is continuous at y, there is $\delta > 0$ such that $|y - y'| < \delta$ implies $|F(y) - F(y')| < \varepsilon/2$. Since $\left|y_{k-1}^n - y_k^n\right| = \overline{y}/n \to 0$, we can choose n_0 such that $\left|y_{k-1}^n - y_k^n\right| < \delta$ for all $n > n_0$. Since $\max\left\{|y - y_k^n|, \left|y - y_{k-1}^n\right|\right\} \leq \left|y_{k-1}^n - y_k^n\right|$, we have that for all $n > n_0$,

$$\begin{aligned} |F_n(y) - F(y)| &\leq \left|F(y_k^n) - F(y_{k-1}^n)\right| \\ &\leq |F(y_k^n) - F(y)| + \left|F(y) - F(y_{k-1}^n)\right| \\ &\leq \varepsilon \end{aligned}$$

which is what we wanted to show. The proof that $G_n \to G$ is analogous and is left to the reader. Q.E.D.

6 Conclusions

We have axiomatically characterized the Theil ordering of income inequality. In addition to the uncontroversial axioms of anonymity, homogeneity, replication invariance, strong directedness and a standard continuity property, we appealed to an independence and to a decomposability axioms. These two axioms are ordinal implications of Theil Decomposability, the central axiom in Bourguignon [7] and Foster [15] in their characterization of the Theil index. To the best of our knowledge, this is the first fully ordinal characterization of this index.

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