The Core of Economies with Asymmetric Information: An Axiomatic Approach*

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Abstract: *We propose two generalizations of the Davis and Maschler (1965) reduced game property to economies with asymmetric information and apply them in the characterization of two solution concepts. One is Wilson's (1978) Coarse Core and the other is a subsolution of it which we call the Coarse+ Core.*

1 Introduction

Although the Core of an economy with perfect information is a well-established concept that seems to raise no controversy, its generalization to economies where agents have different information seems to be more elusive. Indeed, since Wilson's (1978) seminal paper, a large literature has developed that proposes several different Core concepts for economies with asymmetric information. Wilson (1978), for instance, defines the *Coarse* Core and the *Fine* Core, Yannelis (1991), Allen (1992), Koutsougeras and Yannelis (1993) and Hahn and Yannelis (1995) apply, among others, the *Private* Core and the *Weak Fine* Core, and Vohra (1997) introduces the *Incentive Compatible* Core. This proliferation of Core concepts points to a difficulty in reaching a consensus as to what a sensible generalization of the Core of perfect information economies is. If we look carefully, however, we shall see that there are not too many Core concepts. Many of the different cores of an economy with asymmetric information differ in the economy itself rather than in the way allocations are improved upon, and while differences in the underlying economies can lead to different Core allocations, they cannot lead to differences in the definition of the Core. Thus, Wilson's (1978) Coarse Core and Vohra's (1997) Incentive Compatible Core do not differ in the way allocations are improved upon, but in the way allocations are defined, namely in what coalitions can do. While Wilson (1978) takes into account only physical constraints, Vohra (1997) imposes incentive constraints as well. Similarly, the Weak Fine Core and the Private Core differ only in the definition of feasible allocations. They differ in *what* coalitions can do but not in *how* they improve upon with what they can do.

Nevertheless, even controlling for the different sets of feasible allocations, several definitions of Core concepts remain. It should be stressed that the differences do not lie in the timing of

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the agents' evaluation of different allocations. If the agents evaluate the desirability of different bundles before they get any information, the resulting Core concept will be an *ex-ante* concept. If, on the other hand, the agents use their private information while evaluating bundles, the Core concept will be an *interim* concept. But even restricting attention to the interim phase, the phase where agents do have asymmetric information, we find no consensus about the definition of the core.

There are several ways to justify a given solution concept. One may justify it by its intuitive appeal, one may show an interesting class of economies where the concept is non-empty, or one may show that the concept satisfies nice properties. In this paper we follow the axiomatic approach. Instead of trying to generalize the Core directly to economies with asymmetric information, we generalize some properties that characterize the Core in the context of perfect information economies and see if they characterize some solution concept in the context of asymmetric information, and if so, we ask which concept they characterize. In particular, we choose to follow the lines of Peleg's (1985) axiomatization of the Core of cooperative games without side payments. There the axioms used are Individual rationality, Non-emptiness, Consistency and Converse Consistency. With a slight modification of the axioms, Serrano and Volij (1997) show that the Core of an economy with perfect information is characterized by One Person Rationality, Consistency and Converse Consistency. As it is well-known, the consistency property depends on the way the reduced game is defined. Peleg (1985), Serrano and Volij (1997) and many others apply the reduced game that is inspired by Davis and Maschler (1965). According to this approach, a coalition in a reduced economy with respect to a status quo can obtain the cooperation of agents outside the reduced economy by compensating them with bundles that are more attractive than the status quo ones.[1] When we want to adapt the Davis-Maschler reduced game to economies with asymmetric information, we find two equally appealing ways to do so, appealing in the sense that they preserve the usual interpretation of the reduced economy. They differ in the way the individuals in the reduced economy compensate the individuals outside it for their cooperation. The compensation should be attractive enough for the individuals outside the reduced economy to be willing to cooperate but there are at least two ways to make a proposal acceptable. One is to make a proposal that is common knowledge among the proposers and proposees that it is beneficial to all the parties involved. That is, the proposal should remain attractive even after learning that everybody agrees to its terms, that everybody knows that everybody agrees to its terms, and so on. Another way to make an acceptable proposal is to propose something that dominates the status-quo, namely that remains attractive no matter what can be learned from it. Proposals like this cannot be refused. Our findings are as follows. When we follow the first way to make acceptable proposals in order to define the reduced economy, it turns out that Wilson's (1978) Coarse Core is characterized by the axioms of One Person Rationality, Consistency and Converse Consistency. When, on the other hand, we use proposals that cannot be refused in order to define the reduced economy, we get that the same three axioms, are not enough to characterize a solution concept. Adding the axiom of Weak Efficiency, however, suffices to characterize a Core concept that has not been defined before, and that we call the Coarse+ Core. In the case of economies with perfect information, One Person Rationality and Consistency imply Weak Efficiency. Consequently, this last axiom, though satisfied by the Core, is not required. When we deal with economies with asymmetric information, and when getting cooperation from someone requires proposals that cannot be refused, One Person Rationality, Consistency and Weak Efficiency become independent axioms, and all of them are required to get a tight characterization of the Core. It turns out that this second Core concept is a subset of the Coarse Core.

The remainder of this paper is organized as follows. Section 2 presents some basic definitions

[1] For reduced economies that follow a different approach, see for example van den Nouweland, Peleg, and Tijs (1996).

regarding the agents that compose an economy with asymmetric information. Section 3 reviews several definitions of allocations that have appeared in the literature and further discusses alternative notions of improving coalitions. It ends with the definition of the associated Core concepts. Section 4 presents a characterization of two solution concepts. One is Wilson's (1978) Coarse Core and the other is a subsolution of it which we call the Coarse+ Core. Both axiomatizations are obtained by appropriately interpreting and generalizing the reduced economies that are behind the consistency properties that characterize the core of economies with perfect information.

2 Preliminaries

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let U be a set of names. Elements of $\mathcal{F}$ are called events.

Definition 1 An **agent** $i \in U$ is a fourtuple $(X_i, \mathcal{F}_i, u_i, e_i)$ where:

$X_i \subseteq \mathbb{R}^l$ is i's consumption set

$\mathcal{F}_i \subseteq \mathcal{F}$ is a σ-algebra that represents i's information

$u_i : X_i \times \Omega \to \mathbb{R}$ is agent i's state contingent utility function

$e_i : \Omega \to \mathbb{R}^l$ is an $\mathcal{F}$-measurable function that represents agent i's state contingent initial endowment of commodities.

A *bundle* for agent i is a measurable function $x_i : \Omega \to X_i$ that assigns a commodity vector to each state of the world. We denote the set of bundles for i by $\mathbb{B}_i$. For each bundle $x_i \in \mathbb{B}_i$ we denote by $u_i(x_i)$ the function $u_i(x_i) : \Omega \to \mathbb{R}$ such that $u_i(x_i)(\omega) = u_i(x_i(\omega), \omega)$. We shall assume that for each bundle $x_i \in \mathbb{B}_i$, $u_i(x_i)$ is Lebesgue-integrable. Let $x_i : \Omega \to X_i$ be a bundle for i. Agent i's conditional expected utility of x_i relative to $\mathcal{F}_i$ is an $\mathcal{F}_i$-measurable function $\mathrm{E}[u_i(x_i)|\mathcal{F}_i] : \Omega \to \mathbb{R}$ such that:

$$\int_B \mathrm{E}[u_i(x_i)|\mathcal{F}_i]\, d\mu = \int_B \mathrm{E}[u_i(x_i)]\, d\mu \qquad \forall B \in \mathcal{F}_i.$$

Finite subsets of U are called coalitions. Let N be a coalition, let E be an event and let $i \in N$ be an agent. We say that i *knows* E at state ω if there is and event $C \in \mathcal{F}_i$ that i can discern, such that $\omega \in C \subseteq E$. We say that the event E is *common knowledge at* ω among the members of N if there is an event $C \in \cap_{k\in N}\mathcal{F}_k$ that all of them can discern, such that $\omega \in C \subseteq E$. Given two measurable functions $f : \Omega \to \mathbb{R}$ and $g : \Omega \to \mathbb{R}$, we say that an agent knows that $f > g$, if he knows the event $\{\omega : f(\omega) > g(\omega)\}$. When a σ-algebra $\mathcal{F}_i$ is generated by a measurable partition of the state space Ω, we denote by $\mathcal{PF}_i$ the partition of Ω that generates $\mathcal{F}_i$ and we write $\mathcal{PF}_i(\omega)$ for the element of the partition that contains state ω. We denote the finest common coarsening of the partitions $(\mathcal{PF}_i)_{i\in N}$ by $\mathcal{PF}$.

Definition 2 **A pre-economy with asymmetric information**, $(X_i, \mathcal{F}_i, u_i, e_i)_{i\in N}$, is a finite collection of agents.

There are several assumptions that can be made about the pre-economy, some of which, for example, common prior, finite dimensional commodity space, etc., have already been implicitly made. We shall enumerate some conditions that can possibly be assumed on the pre-economy.

Alternative Assumptions on the Pre-Economy

Assumptions on the measure space

- The measure space $(\Omega, \mathcal{F}, \mu)$ is finite and μ assigns positive probability to each of the states of the world.
- The measure space $(\Omega, \mathcal{F}, \mu)$ is infinite, and μ is σ-additive.
- Ω is a product set, namely $\Omega = \prod_{i \in N} T_i$ where T_i is the set of agent i's types.

Assumptions on the utility functions u_i.

- $u_i(\cdot, \omega)$ is concave for all $\omega \in \Omega$.s
- $u_i(\cdot, \omega)$ is strictly increasing for all $\omega \in \Omega$.

Assumptions on the endowments

- For each $i \in N$, e_i is measurable with respect to $\mathcal{F}_i$.
- For each $i \in N$, $e_i \gg 0$.

Assumptions on the information fields

- For each $i \in N$, $\mathcal{F}_i$ generates a countable partition of Ω where each partition cell has positive measure.
- For each $i \in N$, the σ-algebra generated by $\cup_{k \neq i} \mathcal{F}_k$ is $\mathcal{F}$.
- For each $i \in N$, $\mathcal{F}_i$ is the σ-algebra generated by

$$\left(\{(t_1, \ldots, t_n) \in \prod_{k \in N} T_k : t_i = s_i\} \right)_{s_i \in T_i}$$

No matter which assumptions we choose to make about its components, a pre-economy is a description of the individuals that compose it and of the uncertainty they face. In particular, the description of the pre-economy does not tell anything about the activities the agents can engage in, the kind of contracts they can sign, or when those activities or contracts take place and are carried out. Furthermore, the description of the pre-economy does not tell us about the possibilities of exchange of information the agents have. Instead, a pre-economy is just a description of the agents' characteristics or more generally, a collection of agents facing some common background uncertainty.

3 Towards a Definition of the Core

3.1 Allocations

In order to define an economy, it is essential to complement the individuals' characteristics with a description of what they can do. We can think of this as the rules of the game. Without this set of feasible outcomes, we cannot even start to predict the physical outcome of a pre-economy. This description of what different coalitions can do, is summarized by the concept of an S-allocation. Intuitively, an S-allocation is the set of bundles that the coalition S can guarantee for themselves. For a coalition S, we denote by $\mathcal{A}(S)$ the set all its S-allocations. A typical member of $\mathcal{A}(S)$ is a function $y : S \to \cup_{i \in S} \mathbf{B}_i$ such that $y(i) \in \mathbf{B}_i, \forall i \in S$. N-allocations are simply called allocations. Note that the set of S-allocations is a general enough concept to allow for production possibilities or for costs of forming coalitions. An S-allocation can be interpreted as a distribution of bundles among the members of S that they can carry out *without* the consent of the other members of the pre-economy. If y is an S-allocation and $T \subseteq S$, we write y_T for the projection of y on T and, in particular, y_i for $y(i)$.

Different definitions of S-allocations correspond to different social arrangements or social norms. Below we give a list of some possible definitions of S-allocations, some of which have appeared in the literature. (The relevant papers appear in parenthesis.)

Alternative definitions of S-allocations

- (Allen (1992), Yannelis (1991), Koutsougeras and Yannelis (1993), Hahn and Yannelis (1995)) An S-*allocation* is a collection of bundles $y : S \to \cup_{i\in S}\mathbb{B}_i$ with $y_i \in \mathbb{B}_i$, such that:
  1. $\sum_{i\in S} y_i = \sum_{i\in S} e_i$,
  2. For all $i \in S$, y_i is $\mathcal{F}_i$-measurable.
- (Allen (1994)) An S-*allocation* is a collection of bundles $y : S \to \cup_{i\in S}\mathbb{B}_i$ with $y_i \in \mathbb{B}_i$, such that:
  1. $\sum_{i\in S} y_i = \sum_{i\in S} e_i$
  2. For all $i \in S$, $y_i - e_i$ is measurable with respect to an exogenously given σ-field $\hat{f}^i(S)((\mathcal{F}_i)_{i\in S})$ (note that the exogenous σ-field depends on the coalition).
- (Koutsougeras and Yannelis (1993)) An S-*allocation* is a collection of bundles $y : S \to \cup_{i\in S}\mathbb{B}_i$ with $y_i \in \mathbb{B}_i$, such that:
  1. $\sum_{i\in S} y_i = \sum_{i\in S} e_i$
  2. For all $i \in S$, y_i is measurable with respect to the σ-algebra generated by $\cup_{k\in S}\mathcal{F}_k$.
- (Koutsougeras and Yannelis (1993)) An S-*allocation* is a collection of bundles $y : S \to \cup_{i\in S}\mathbb{B}_i$ with $y_i \in \mathbb{B}_i$, such that:
  1. $\sum_{i\in S} y_i = \sum_{i\in S} e_i$,
  2. For all $i \in S$, $y_i - e_i$ is $\cap_{k\in S}\mathcal{F}_k$-measurable.[2]
- (Wilson (1978)) An S-*allocation* is a collection of bundles $y : S \to \cup_{i\in S}\mathbb{B}_i$ with $y_i \in \mathbb{B}_i$, such that:
  1. $\sum_{i\in S} y_i = \sum_{i\in S} e_i$.
- (Vohra (1997)) When the state space is the product of the players' types and the information structure is generated by the types, an S-*allocation* is a collection of bundles $y : S \to \cup_{i\in S}\mathbb{B}_i$ with $y_i \in \mathbb{B}_i$, such that:
  1. $\sum_{i\in S} y_i = \sum_{i\in S} e_i$,
  2. For all $i \in S$, y_i is measurable with respect to the σ-algebra generated by $\cup_{k\in S}\mathcal{F}_k$,
  3. y_S is incentive compatible; namely

     $$\mathrm{E}[u_i(y_i)|\mathcal{F}_i] \geq \mathrm{E}[u_i(y_i(s_i))|\mathcal{F}_i], \quad \forall s_i \in T_i, \quad \forall i \in S$$

     where $y_i(s_i) : \Omega \to \mathbb{B}_i$ such that $y_i(s_i)(t) = y_i(t_{-i}, s_i)$.

[2]In fact, Koutsougeras and Yannelis (1993) impose this measurability restriction only for proper coalitions of N.

Each of the above definitions of S-allocations makes sense in different contexts. The S-allocations used in Vohra (1997), for example, are natural when individuals face not only physical but also incentive constraints. The S-allocations used in Wilson (1978), on the other hand, make sense when the true state is verifiable at the time contracts are fulfilled.

We can now present the definition of an economy.

Definition 3 An **economy** is a pair $\langle \mathcal{PE}, (\mathcal{A}(S))_{S\subseteq N}\rangle$ where $\mathcal{PE} = (X_i, \mathcal{F}_i, u_i, e_i)_{i\in N}$ is a pre-economy and $(\mathcal{A}(S))_{S\subseteq N}$ is the collection of all its S-allocations.

We are interested in economies where allocations are agreed upon in the *interim* phase, namely after each agent knows his own private information but before the realized state is revealed to him. The natural definition of individual rationality in this case is the following.

Definition 4 Let $\langle \mathcal{PE}, (\mathcal{A}(S))_{S\subseteq N}\rangle$ be an economy. An allocation $x \in \mathcal{A}(N)$ is **individually rational** if there is no state ω, agent $i \in N$ and $\{i\}$-allocation $y \in \mathcal{A}(\{i\})$ such that $\mathrm{E}[u_i(y)|\mathcal{F}_i](\omega) > \mathrm{E}[u_i(x_i)|\mathcal{F}_i](\omega)$.

In other words, an allocation is individually rational if there is no agent i, $\{i\}$-allocation $y \in \mathcal{A}(\{i\})$ and state ω at which i knows that he prefers y to x_i. Next, we define weak efficiency as follows:

Definition 5 Let $\langle \mathcal{PE}, (\mathcal{A}(S))_{S\subseteq N}\rangle$ be an economy. An allocation $x \in \mathcal{A}(N)$ is **weakly efficient** if there is no allocation $y \in \mathcal{A}(N)$ and state $\omega \in \Omega$ at which it is common knowledge that $\mathrm{E}[u_i(y_i)|\mathcal{F}_i] > \mathrm{E}[u_i(x_i)|\mathcal{F}_i]$, $\forall i \in N$.

If Ω is the only event that is common knowledge among the members of N, $x \in \mathcal{A}(N)$ is weakly efficient if and only if there is no allocation $y \in \mathcal{A}(N)$ that is preferred by all agents at every state of the world.

3.2 Improving Coalitions

What complicates the analysis of improving coalitions in economies with asymmetric information is that agents are necessarily conscious of the fact that they are improving upon an allocation when they are doing so. However, if and how they update their information based on this is implicit in the particular definition of improving coalitions which one uses. If agents update and refine their information based on the fact that they are improving upon an allocation, they may decide that a *proposed* improving allocation is in fact not desirable.

To circumvent this problem, Wilson (1978) proposed a definition of improving coalitions which does not rely on agents proposing a blocking move over the status quo. Instead, a coalition S improves upon the status quo if and only if it is self evident to them that there exists an alternative S-allocation that each agent in S strictly prefers. Formally:

Definition 6 (Wilson (1978)) A coalition S **(coarsely) improves upon** allocation $x \in \mathcal{A}(N)$ if there exists an S-allocation y, and a state ω at which it is common knowledge among the members of S that:

$$\mathrm{E}[u_i(y_i)|\mathcal{F}_i] > \mathrm{E}[u_i(x_i)|\mathcal{F}_i] \qquad \forall i \in S$$

That is, in order for the coalition S to (coarsely) improve upon the status quo, agents need not communicate in any way, since it is common knowledge that all agents in S are better off from y than they were from x_S. Wilson's coarse definition has also been used by Kobayashi (1980), who applies it to economies with production and by Vohra (1997), who applies it to the set of incentive compatible allocations.

To see how agents improve upon an allocation using Wilson's notion of coarse blocking consider the following example from Wilson (1978) with a single commodity in which the status quo is the endowment.

			Endowments (e_i)			Allocation (y_i)		
Agent	$\mathcal{PF}_i$	State (ω)	ω_1	ω_2	ω_3	ω_1	ω_2	ω_3
1	$\{\omega_1\},\{\omega_2,\omega_3\}$		5	1	3	$5+\varepsilon$	2	$2-\varepsilon$
2	$\{\omega_2\},\{\omega_1,\omega_3\}$		3	5	1	$2-\varepsilon$	$5+\varepsilon$	2
3	$\{\omega_3\},\{\omega_1,\omega_2\}$		1	3	5	2	$2-\varepsilon$	$5+\varepsilon$

Table 1: Example from Wilson (1978)

Suppose that each agent has a constant utility function $u = \ln(a)$ and there exists a common prior $\mu = \left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$. Note that in this example $\{\omega_1,\omega_2,\omega_3\}$ is the only event which is common knowledge among the grand coalition. Then, at any state, it is common knowledge among the members of $S = \{1,2,3\}$ that $\mathrm{E}[u_i(y_i)|\mathcal{F}_i] > \mathrm{E}[u_i(e_i)|\mathcal{F}_i]$, $\forall i \in S$. That is, it is self-evident that the allocation y is preferred by all agents to the endowment. The agents in S need not communicate among one another in order to determine that y is preferred over e; no agent needs to "propose" y and convince them that it is better. To the agents in S, it is self evident that y is preferred to e by all of them, and therefore this fact does not carry any new informational content. Consequently, after taking into account that y is preferred to e, the agents still prefer y to e.

One may think that the common knowledge restriction placed upon coalitions in order to improve upon allocations is too strict, and one might imagine several alternative ways in which coalitions could improve upon the status quo without it being common knowledge that all agents in a coalition are made better off. To this end, Hahn and Yannelis (1995) have suggested the following alternative definition of an improving coalition:

Definition 7 (Hahn and Yannelis (1995)) A coalition S **(naïvely) improves upon** allocation $x \in \mathcal{A}(N)$ if there exists an S-allocation y, and a state ω such that:

$$\mathrm{E}[u_i(y_i)|\mathcal{F}_i](\omega) > \mathrm{E}[u_i(x_i)|\mathcal{F}_i](\omega) \qquad \forall i \in S.$$

That is, a coalition S improves upon the status quo x if there exists an allocation $y \in \mathcal{A}(S)$ and a state ω at which they know that they prefer y to x. We call this improvement naïve because the agents do not update their information based on the fact that they are part of an improving coalition. This point is best illustrated by considering the classic "trading envelopes" example.[3] Consider the economy described in Table 2 with two agents, a single commodity x which we can think of as money, four states of the world, and a common prior $\mu = \left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$. We assume that each agent has a constant across states utility function given by $u(a) = a$.

		Endowment $(e_i)_N$				Allocation y			
Agent	$\mathcal{PF}_i$	ω_1	ω_2	ω_3	ω_4	ω_1	ω_2	ω_3	ω_4
1	$\{\{\omega_1\},\{\omega_2,\omega_3\},\{\omega_4\}\}$	1	4	4	16	2	2	8	8
2	$\{\{\omega_1,\omega_2\},\{\omega_3,\omega_4\}\}$	2	2	8	8	1	4	4	16

Table 2: Trading Envelopes

[3]For a complete analysis of the envelopes example, see Nalebuff (1989) and Geanakoplos (1992).

Suppose the true state of the world is $\omega^* = \omega_3$. Then, agent 1 believes the state is either ω_2 or ω_3 but he is certain that his "envelope" has \$4. Agent 2 believes the state is either ω_3 or ω_4, but she is certain that her envelope contains \$8. Now consider the allocation which is achieved by exchanging endowments, that is, "trading envelopes". Given that the true state of the world is ω_3, agent 1 is better off in expected value from the trade since the expected utility of his envelope is now 5, which is greater than the sure utility of 4 which he received from his endowment. Similarly, agent 2 is better off since she now has an expected utility of 10 instead of the sure utility of 8 which she received from her endowment. That is, the allocation y in Table 2 above naïvely improves upon the endowment. Consider what agent 2 should be able to deduce. At ω_3, she knows only that the state is either ω_3 or ω_4. Furthermore, she knows that agent 1 knows that the state is either in $\{\omega_2, \omega_3\}$ or that the state is ω_4. Obviously agent 2 prefers trading envelopes to keeping her endowment because of the hope that the state is ω_4. However, it is clear that if the state were ω_4, agent 1 would not want to trade envelopes. That is, from the very fact that she is blocking with allocation y, agent 2 should be able to learn that the true state is ω_3. Agents who are in a naïve improving coalition fail to update their prior information based on the "common knowledge" fact that they are blocking. Note that in this example, the agents do not (coarsely) improve upon the endowments by trading envelopes since it cannot be common knowledge that both agents expect a gain.[4]

Although the common knowledge requirement of the coarse improving coalitions avoid the "unsophisticated" types of blocking that the naïve form of blocking is susceptible to, the requirement that it be common knowledge among *all* agents in a coalition to improve upon the status quo is rather demanding. The previous two examples illustrate that the agents in an improving coalition should be able to update their information from the fact that they are part of an improving coalition. However, once one allows for updating, one must accept the possibility that agents can iteratively update their information *ad infinitum*. In order to avoid such complications, we introduce a notion of improving coalitions based on dominant offers. Consider the following definition:

Definition 8 (Lee (1997)) A coalition S **(individualistically) improves upon** allocation $x \in \mathcal{A}(N)$ if there exists an agent $j \in S$, S-allocation $y \in \mathcal{A}(S)$, and state ω such that:

1. $u_i(y_i) > u_i(x_i) \qquad \forall i \in S \setminus \{j\}$,
2. $\mathrm{E}[u_j(y_j)|\mathcal{F}_j](\omega) > \mathrm{E}[u_j(x_j)|\mathcal{F}_j](\omega)$

In this definition, there is one "active" agent, j, who sees an opportunity for personal gain and makes proposals to the remaining members of S which they "cannot refuse."[5] By using dominant proposals, this definition avoids the problems of the naïve definition, since any agent in $S \setminus \{j\}$ is made strictly better off at every state, and the fact that he accepts the proposal reveals nothing to agent j. That is, agents update their information taking into account the fact that they are blocking, but this new information does not cause them to change their behavior. At the same time, the individualistic definition of improving coalitions relaxes the common knowledge requirement

[4]One might argue that the above example is not applicable to the type of blocking suggested by Hahn and Yannelis (1995) since the improving allocation y is not $\mathcal{PF}_i$-measurable. We contend however that a definition of blocking should not be susceptible to perturbations in the definition of the S-allocations. Nevertheless, it is not difficult to find examples with $\mathcal{PF}_i$-measurable S-allocations in which the naïve improving coalition still suffers from this short-sighted behavior.

[5]The reader should be aware that this is a slightly modified version of the definition presented in Lee (1997), since the proposees are required to be left strictly better off in *every* state. Lee (1997) requires that the proposees be left better off in every state which the proposer thinks the proposees think are possible. We have chosen to simplify the definition here for purposes of clarity.

of the coarse definition, since each agent in $\{S \setminus j\}$ need not know anything about the utility of any other agent in S. To illustrate the individualistic notion of improving coalitions, consider the following example of an economy with two agents, two commodities a and b, two states of the world, and a common prior $\mu = \left(\frac{1}{2}, \frac{1}{2}\right)$. Agents' utility functions are constant across states and given by $u(a,b) = \min\{a,b\}$ for both agents and that the S-allocations are those as defined by Wilson (1978).

		Endowment $(e_i)_N$		Allocation y	
Agent	$\mathcal{PF}_i$	ω_1	ω_2	ω_1	ω_2
1	$\{\{\omega_1,\omega_2\}\}$	$(5,0)$	$(5,0)$	$(6,1)$	$(2.5,2.5)$
2	$\{\{\omega_1\},\{\omega_2\}\}$	$(1,1)$	$(0,5)$	$(0,0)$	$(2.5,2.5)$

Table 3

Since Ω is the only common knowledge event, the endowment cannot be coarsely improved upon since there is no way to make agent 2 better off in state ω_1. However, suppose that the state is $\omega^* = \omega_2$. Then, agent 2 can offer the allocation y to agent 1 which he cannot refuse. Under the blocking allocation y, agent 2 is better off in state ω_2 since her utility increases from 0 to 2.5. Agent 1 is strictly better off in all states: y_1 is a bundle which agent 1 cannot refuse. Note that from the proposal that agent 2 makes, agent 1 can in fact infer that the true state must be ω_2. However, this information does not change agent 1's optimal behavior since y_1 is a bundle which makes him better off in all states. Furthermore, since the proposal was strictly dominant, the fact that agent 1 accepts her proposal reveals nothing to agent 2.

The example from Table 3 shows that the individualistic definition of improving coalitions can improve upon allocations which the coarse definition cannot. However, it places the restriction that there can only be one "active" agent and there exist simple economies in which one active agent may not be able to initiate a successful improving coalition, whereas two or more can. Consider the following economy with four states of the world and three commodities a, b, and c. There is a common prior given by $\mu = (.25, .25, .25, .25)$ and the constant across states utility function for all agents is $u(a,b,c) = \min\{a,b,c\}$. The consumption set for each agent is $X_i = \mathbb{R}^3_+$ and the S-allocations are given by those as defined by Wilson (1978).

		Endowment $(e_i)_N$			
Agent	$\mathcal{PF}_i$	ω_1	ω_2	ω_3	ω_4
1	$\{\{\omega_1,\omega_2\},\{\omega_3\},\{\omega_4\}\}$	$(1,0,0)$	$(1,0,0)$	$(1,0,0)$	$(1,1,1)$
2	$\{\{\omega_1\},\{\omega_2,\omega_3\},\{\omega_4\}\}$	$(0,1,0)$	$(0,1,0)$	$(0,1,0)$	$(1,1,1)$
3	$\{\{\omega_1,\omega_2,\omega_3,\omega_4\}\}$	$(0,0,1)$	$(0,0,1)$	$(0,0,1)$	$(0,0,1)$

Table 4

It is not difficult to check that the endowment cannot be (coarsely) improved upon by any coalition. It is clear also that no coalition can individualistically improve upon the endowment since there is no agent who, by himself, can make any other agent better off in states ω_1, ω_2 and ω_3.

We now show that it is possible for the agents to improve upon the endowment when agents 1 and 2 are active, and agent 3 is passive. Suppose that $\omega^* = \omega_1, \omega_2$ or ω_3 and consider the following allocation:

		Allocation y			
Agent	$\mathcal{PF}_i$	ω_1	ω_2	ω_3	ω_4
1	$\{\{\omega_1,\omega_2\},\{\omega_3\},\{\omega_4\}\}$	$\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$	$\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$	$\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$	$(0,0,0)$
2	$\{\{\omega_1\},\{\omega_2,\omega_3\},\{\omega_4\}\}$	$\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$	$\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$	$\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$	$(0,0,0)$
3	$\{\{\omega_1,\omega_2,\omega_3,\omega_4\}\}$	$\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$	$\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$	$\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$	$(2,2,3)$

Table 5

Agent 3 is better off in every state from this allocation compared to the endowment. Since the true state is ω_2, ω_3 or ω_4, the event $\{\omega_1, \omega_2, \omega_3\}$ is common knowledge between agents 1 and 2, and it is clear that they are both better off in this event under the proposed allocation.

We now define a generalized version of the individualistic improving coalition which allows for coalitions to make dominant offers.

Definition 9 A coalition S **(Coarsely+) improves upon** an allocation $x \in \mathcal{A}(N)$ if there exists an S-allocation y, a state ω and a partition $\{A, P\}$ of S such that:

1. $u_i(y_i) > u_i(x_i), \quad \forall i \in P$,
2. It is common knowledge at ω among the members of A that

$$\mathrm{E}[u_i(y_i)|\mathcal{F}_i] > \mathrm{E}[u_i(x_i)|\mathcal{F}_i] \qquad \forall i \in A$$

According to the above definition, an allocation x can be improved upon by a coalition S, if there is a subgroup A of active agents that can make a proposal to the remaining agents, P, that cannot be refused and that is commonly known by the members of A that is preferred to x.

The Coarse+ notion of improving coalitions lies between the individualistic and coarse definitions. Note that when $P = \emptyset$, the Coarse+ definition is identical to the coarse definition of Wilson (1978) (Definition 6). However, when A is a singleton, the Coarse+ definition coincides with the individualistic definition (Definition 8).[6] We call this notion of improving Coarse+ since it uses the common knowledge requirement of the coarse definition, but adds the possibility of using dominant offers in order to improve upon the status quo.

3.3 Solutions

Now that several notions of improving coalitions have been defined, we can proceed to give several alternative definitions of the Core for economies with asymmetric information. Any definition of the Core is an example of a solution concept for a class of economies and hence we begin by defining what a solution is.

Definition 10 Let E be a class of economies. A **solution on** E is a set-valued function φ that assigns to each economy $\langle \mathcal{PE}, (\mathcal{A}(S))_{S \subseteq N} \rangle \in \mathsf{E}$ a set of allocations in $\mathcal{A}(N)$.

For example,

- The **Empty solution** assigns to each economy in E the empty set.
- The **Pareto optimal solution** ($\mathcal{PO}$) assigns to each economy in E the set of its weakly efficient allocations.
- The **Individually Rational** ($\mathcal{IR}$) solution assigns to each economy in E the set of its individually rational allocations.
- The **Coarse Core** ($\mathcal{CC}$) assigns to each economy in E the set of the allocations that are *(coarsely) improved upon* by no coalition.
- The **Individualistic Core** ($\mathcal{IC}$) assigns to each economy in E the set of the allocations that are *(individualistically) improved upon* by no coalition.

[6] Note that if $x \in \mathcal{A}(N)$ can be coarsely+ improved upon, then $A \neq \emptyset$ since Ω is common knowledge at every state.

- The **Coarse+ Core** ($\mathcal{C}^+$) assigns to each economy in E the set of the allocations that are *(coarsely+) improved upon* by no coalition.

4 An Axiomatic Approach

The reason why there are many alternative definitions of the improving coalitions and of the Core of an economy with asymmetric information is that there are several appealing ways to generalize the definition of the Core of an economy with perfect information. Recall that an allocation x in an economy with perfect information is improved upon by a coalition S if there is an S-allocation that provides each member of S with a higher utility than the allocation x. Formally, coalition S improves upon allocation $x \in \mathcal{A}(N)$ if there is an S-allocation $y \in \mathcal{A}(S)$ such that $u_i(y_i) > u_i(x_i)$ for all $i \in S$. This formal definition can be interpreted in two ways. One interpretation is that it is self-evident to the members of S that the S-allocation is preferred by all of them to the status quo. In other words, nobody needs to point out that y is preferred to x_S for them to realize so. Namely it is common knowledge among the members of S that all of them prefer y to x. The second interpretation is that there is a member i of S that can make $S \setminus \{i\}$ a proposal that they "cannot refuse," and that if accepted, is beneficial to i. The proposal would be phrased as: "Give me your endowments and I will give you y in return." Note that it is dominant for the proposees to accept the offer and thus it is truly one that $S \setminus \{i\}$ cannot refuse. Furthermore, for the coalition to improve upon the status quo, the proposer must also be left better off from the deal. More generally, we can interpret the formal definition above as a subgroup of S (not necessarily a single individual) making the remaining members a proposal that they "cannot refuse" and that, when accepted, makes each one of the proposers better off. According to this second interpretation, there is some communication among the agents in the economy, but it is restricted to proposals that cannot be refused. While these two interpretations of an improving coalition lead to the same formal definition in economies with perfect information, they diverge when we deal with economies with asymmetric information.

In this section we follow an axiomatic approach. Instead of generalizing the well-established concept of the Core in a direct way, we generalize the axioms that characterize it on the class of economies with perfect information and then ask if these generalized axioms characterize a solution on the class of economies with asymmetric information. If the answer is in the affirmative, we can then see what is the solution concept associated with the axioms.

Our starting point is the axiomatization of the Core of cooperative games without transferable utility due to Peleg (1985), which is based on Consistency, Converse Consistency, Individual rationality and Non-emptiness. Some properties don't seem to be controversial in the way they are generalized. This is the case of Individual Rationality, for instance, or of efficiency. Generalizations of the reduced economies that are behind the consistency axioms, however, seem less straightforward. In this section we pursue two generalizations that follow from the above two interpretations of improving allocations, respectively.

4.1 Axiomatization of the Coarse Core

In this subsection we provide an axiomatization of Wilson's (1978) Coarse Core, using the consistency principle.[7] In order to apply this principle we need to define the appropriate notion of a reduced economy with respect to a coalition S and a status-quo $x \in \mathcal{A}(N)$.

The agents that compose this reduced economy are the members of S, and the F-allocations that each sub-coalition F of S can enforce are built as follows. If F is a strict subset of S, then

[7] For a comprehensive surveys on consistency and its applications, the reader is referred to Thomson (1995).

they can get the cooperation of any non-empty coalition $G \subseteq N \setminus S$ by enforcing with them any $S \cup G$-allocation that is commonly known by $S \cup G$ to be strictly preferred by them. Note that coalition F can perform this operation without the consent of the other members of the reduced economy. If F chooses not to cooperate with any outside coalition they can still achieve any F-allocation in $\mathcal{A}(F)$. If $F = S$, they can get the cooperation of any coalition outside S in the same way just described, and add them to the set of S-allocations $\mathcal{A}(S)$ that they can enforce without cooperation, but in addition, they can enforce the status-quo x as well.[8]

In order to define the reduced economy, we need the following piece of notation. Let S be a coalition and let $x \in \mathcal{A}(N)$ be an allocation. The set of S-allocations that are commonly known among the member of S at some state to be strictly preferred to x is:

$$\mathcal{P}^x(S) = \left\{ y \in \mathcal{A}(S) \middle| \begin{array}{l} \text{there is a state } \omega \in \Omega \text{ at which it is common knowledge} \\ \text{among } S \text{ that } \mathrm{E}[u_i(y_i)|\mathcal{F}_i] > \mathrm{E}[u_i(x_i)|\mathcal{F}_i],\ \forall i \in S. \end{array} \right\}$$

The reduced economy is formally defined as follows.

Definition 11 Let $\mathcal{E} = \langle (X_i, \mathcal{F}_i, u_i, e_i)_{i \in N}, (\mathcal{A}(S))_{S \subseteq N} \rangle$ be an economy, let $x \in \mathcal{A}(N)$ be an allocation and let S be a coalition. The **reduced economy** with respect to x and S is defined as:

$$\mathcal{E}^{S,x} = \langle (X_i, \mathcal{F}_i, u_i, e_i)_{i \in S}, (\mathcal{A}^{S,x}(F))_{F \subseteq S} \rangle$$

where

$$\mathcal{A}^{S,x}(F) = \begin{cases} \bigcup_{G \subseteq N \setminus S} \{y_F : y \in \mathcal{P}^x(F \cup G)\} \cup \mathcal{A}(F) \cup \{x_S\} & \text{if } F = S \\ \bigcup_{G \subseteq N \setminus S} \{y_F : y \in \mathcal{P}^x(F \cup G)\} \cup \mathcal{A}(F) & \text{if } F \subsetneq S \end{cases}$$

It is interesting to check which assumptions on the economy are preserved under the reduction operation. Clearly, any assumption on the agents is preserved, since the reduced economy modifies only the set of S-allocations. Consequently, only the assumptions on the S-allocations need be checked.

Consider for example an economy $\langle (X_i, \mathcal{F}_i, u_i, e_i)_{i \in N}, (\mathcal{A}(S))_{S \subseteq N} \rangle$ where for all $S \subseteq N$, and for all $i \in S$, if $y \in \mathcal{A}(S)$ then y_i is $\mathcal{F}_i$ measurable. Let $x \in \mathcal{A}(N)$, $S \subseteq N$ and $F \subseteq S$. Allocations in the reduced economy with respect to x and S are composed of bundles that are measurable with respect to the respective agents' information. To see this, note that if $y_F \in \mathcal{A}^{S,x}(F)$ then by the definition of the reduced economy there is a subset G of $N \setminus S$ and an $(F \cup G)$-allocation $y \in \mathcal{A}(F \cup G)$ such that y_F is the projection of y on F, and by assumption y_i is $\mathcal{F}_i$-measurable for all $i \in F \cup G$.

Similarly, consider an economy $\langle (X_i, \mathcal{F}_i, u_i, e_i)_{i \in N}, (\mathcal{A}(S))_{S \subseteq N} \rangle$ where for all $S \subseteq N$, and for all $i \in S$, if $y \in \mathcal{A}(S)$ then $y_i - e_i$ is $\cap_{k \in S} \mathcal{F}_i$ measurable. Allocations in the reduced economy with respect to x and S are composed of bundles that result from exchanges which are measurable with respect to the information that is common knowledge among S. To see this, let $x \in \mathcal{A}(N)$, $S \subseteq N$ and $F \subseteq N$. If $y_F \in \mathcal{A}^{S,x}(F)$ then by the definition of the reduced economy there is a subset G of $N \setminus S$ and an $(F \cup G)$-allocation $y \in \mathcal{A}(F \cup G)$ such that y_F is the projection of y on F, and by assumption $y_i - e_i$ is $\cap_{k \in F \cup G} \mathcal{F}_k$-measurable for all $i \in F \cup G$. In particular, $y_i - e_i$ is $\cap_{k \in F} \mathcal{F}_k$-measurable for all $i \in F$.

As another important example, consider an economy $\langle (X_i, \mathcal{F}_i, u_i, e_i)_{i \in N}, (\mathcal{A}(S))_{S \subseteq N} \rangle$ where for all $S \subseteq N$, and for all S, $\mathcal{A}(S)$ consists of incentive compatible allocations. Let $x \in \mathcal{A}(N)$, $S \subseteq N$

[8] This asymmetry is necessary if we want to define the Core, as it is usually done, with strict inequalities. Peleg (1985) does not have this asymmetry because it analyzes a class of games where set of efficient and weakly efficient S-allocations coincide and Serrano and Volij (1997) does not have this asymmetry because and the Core is defined with weak inequalities.

and $F \subseteq S$. Allocations in the reduced economy with respect to x and S are incentive compatible as well. To see this note that if $y_F \in \mathcal{A}^{S,x}(F)$ then by the definition of the reduced economy there is a subset G of $N \setminus S$ and an $(F \cup G)$-allocation $y \in \mathcal{A}(F \cup G)$ such that y_F is the projection of y on F. But since y is incentive compatible we have that in particular $\mathrm{E}[u_i(y_i)|\mathcal{F}_i] \geq \mathrm{E}[u_i(y_i(s_i))|\mathcal{F}_i]$, for all $s_i \in T_i$ and for all $i \in F$ which means that y_F is incentive compatible.

Finally, consider on the other hand, the class of economies $\langle (X_i, \mathcal{F}_i, u_i, e_i)_{i \in N}, (\mathcal{A}(S))_{S \subseteq N} \rangle$ where for all $S \subseteq N$, and for all $i \in S$, if $y \in \mathcal{A}(S)$ then y_i is measurable with respect to the σ-algebra generated by $\cup_{k \in S} \mathcal{F}_k$. This class is not closed under the reduction operation. This can be checked after noting that if a bundle y_i is measurable with respect to the σ-algebra generated by $\cup_{k \in F \cup G} \mathcal{F}_k$-measurable, it is not necessarily $\mathcal{F}_i$-measurable.

For our axiomatization it is important to deal with classes of economies that are closed under the reduction operation. These classes are defined as follows.

Definition 12 A class E of economies is said to be **closed under the reduction operation (closed)** if for every $\mathcal{E} \in \mathsf{E}$ with set of agents N, for every coalition $S \subseteq N, S \neq \emptyset$ and for every allocation $x \in \mathcal{A}(N)$, $\mathcal{E}^{S,x} \in \mathsf{E}$.

4.1.1 The Axioms

Now that the reduced economy is defined, we can state the properties that characterize the Core in the class of perfect information economies. The first axiom is very mild and requires that for economies with only one agent, the solution should recommend an individually rational allocation.

Axiom 1 (OPIR) A solution φ on a class E of economies satisfies **one person individual rationality** if it assigns to each one-person economy a subset of its individually rational allocations.

The next axiom is a little bit stronger than OPIR, in that it requires the solution to assign to one-person economies the set of all its individually rational allocations. It was introduced in Peleg and Tijs (1996) in the context of games in strategic form, and used by Serrano and Volij (1997) in the context of perfect information economies.

Axiom 2 (OPR) A solution φ on a class E of economies satisfies **one person rationality** if it assigns to each one-person economy the set of its individually rational allocations.

The next property requires that the solution not recommend weakly inefficient allocations.

Axiom 3 (PAR) A solution φ on a class E of economies satisfies **weak Pareto optimality** if it assigns to each economy a subset of its weakly efficient allocations.

The next axiom is consistency. In our context it requires that if a solution recommends a certain allocation x for an economy with agents set N, it should also recommend the projection of the allocation on $S \subseteq N$ for the reduced economy with respect to S and status-quo x.

Axiom 4 (CONS) A solution φ on a class E of economies satisfies **consistency** if for every $\mathcal{E} \in \mathsf{E}$ with set of agents N, for every $S \subseteq N, S \neq \emptyset$, $x \in \varphi(\mathcal{E})$, implies $x_S \in \varphi(\mathcal{E}^{S,x})$.

The next axiom is a converse of the previous one. It requires from weakly efficient allocations that if their projections on each proper coalition of the set of agents belong to the solution of the corresponding reduced economy, then the allocation x itself should belong to the solution of the economy.

Axiom 5 (COCONS) A solution φ on a class E of economies satisfies **converse consistency** if the following holds. Let $\mathcal{E}$ be an economy in E, and x be a weakly efficient allocation in $\mathcal{E}$. If $x_S \in \varphi(\mathcal{E}^{S,x})$, for all $S \subseteq N, S \neq \emptyset, S \neq N$, then $x \in \varphi(\mathcal{E})$.

It is known (see Peleg (1985) and Serrano and Volij (1997)) that the Core satisfies all the above axioms on the class of economies with perfect information.[9] Next we show that the Coarse Core satisfies these axioms on the class of economies with asymmetric information.

Lemma 1 Let E be a closed class of economies. The Coarse Core satisfies OPR, OPIR, CONS and COCONS on E.

Proof: We show that the Coarse Core satisfies each of the four axioms in order.

OPR For any one-person economy, the coarse Core coincides with the set of individually rational allocations: when restricted to one agent, common knowledge coincides with the knowledge.

OPIR This follows from the fact that the Coarse Core satisfies OPR.

CONS Let $x \in \mathcal{A}(N)$ be an allocation and assume that $x_S \notin \mathcal{CC}(\mathcal{E}^{S,x})$. This means that there is a coalition $F \subseteq S$ that coarsely improves upon x_S. In other words, there is an F-allocation $y_F \in \mathcal{A}^{S,x}(F)$ and state ω such that it is common knowledge among the members of F that

$$\mathrm{E}[u_i(y_i)|\mathcal{F}_i] > \mathrm{E}[u_i(x_i)|\mathcal{F}_i] \qquad \forall i \in F.$$

Since $y_F \in \mathcal{A}^{S,x}(F)$, either $y_F \in \mathcal{A}(F)$, in which case $x \notin \mathcal{CC}(\mathcal{E})$, or there is a coalition $G \subseteq N \setminus S$ and $F \cup G$-allocation $y \in \mathcal{A}(F \cup G)$ such that

1. y_F is the projection of y on F
2. It is common knowledge at some ω among the members of $G \cup F$ that

$$\mathrm{E}[u_i(y_i)|\mathcal{F}_i] > \mathrm{E}[u_i(x_i)|\mathcal{F}_i], \qquad \forall i \in F \cup G.$$

But this implies that $x \notin \mathcal{CC}(\mathcal{E})$, since $F \cup G$ improves upon x.

COCONS Assume that $x \notin \mathcal{CC}(\mathcal{E})$. Then, there is a coalition S that coarsely improves upon x. This means that there exists a state ω^*, and an S-allocation $y \in \mathcal{A}(S)$ such that:

it is common knowledge among the members of S that

$$\mathrm{E}[u_i(y_i)|\mathcal{F}_i] > \mathrm{E}[u_i(x_i)|\mathcal{F}_i], \qquad \forall i \in S. \qquad (**)$$

If $S = N$, then x is not weakly efficient. If $S \subsetneq N$, consider the reduced economy $\mathcal{E}^{S,x}$. Since $y \in \mathcal{A}(S)$, $y \in \mathcal{A}^{S,x}(S)$. By definition of reduced economy $x_S \in \mathcal{A}^{S,x}(S)$ as well. Consequently, by $(**)$ again, $x_S \notin \mathcal{CC}(\mathcal{E}^{S,x})$. □

The following lemma shows that with the present definition of a reduced economy, Individual Rationality and consistency implies Weak Efficiency.

[9] The Coarse Core and the Core coincide on this class.

Lemma 2 Let φ be a solution that satisfies OPIR and CONS. Then φ satisfies PAR.

Proof: Let $x \in \varphi(\mathcal{E})$ and let $i \in N$. By consistency of φ we have $x_i \in \varphi(\mathcal{E}^{\{i\},x})$. Since φ satisfies OPIR, x_i is individually rational, namely there is no $y_i \in \mathcal{A}^{(\{i\},x)}$ that is preferred by i at some $\omega \in \Omega$. This means that there is no allocation $y \in \mathcal{A}(N)$ and state $\omega \in \Omega$ at which it is common knowledge among the members of N that

$$\mathrm{E}[u_j(y_j)|\mathcal{F}_j] > \mathrm{E}[u_j(x_j)|\mathcal{F}_j], \qquad \forall j \in N$$

and such that y_i is the projection of y on $\{i\}$. But this means that x is weakly efficient. □

The following simple and powerful lemma will allow us to characterize the Coarse Core.

Lemma 3 Let φ be a CONS and OPIR solution on E and let ψ be a converse consistent solution on the same class of economies. If $\varphi(\mathcal{E}) \subseteq \psi(\mathcal{E})$ for all one-person economies $\mathcal{E}$, then $\varphi(\mathcal{E}) \subseteq \psi(\mathcal{E}), \forall \mathcal{E} \in \mathsf{E}$.

Proof: The proof is by induction. The claim is trivially true for one-person economies. Suppose now that the statement of Lemma 3, holds for all k-person economies, $1 \leq k \leq n-1$ and let $\mathcal{E} \in \mathsf{E}$ be an n person economy. Let $x \in \varphi(\mathcal{E})$. Since φ satisfies OPIR and CONS, by Lemma 2 x is weakly efficient. By CONS of $\varphi, x_F \in \varphi(\mathcal{E}^{F,x})$ for all $F \subseteq N, N \neq \emptyset, F \neq N$. By the induction hypothesis, $x_F \in \psi(\mathcal{E}^{F,x})$ for all $F \subseteq N, F \neq \emptyset, F \neq N$. Since ψ is converse consistent $x \in \psi(\mathcal{E})$. □

As a corollary of the above result, we learn that the Coarse Core is the solution concept which is maximal with respect to set inclusion, among those that satisfy OPIR and CONS.

Theorem 1 *Let E be a closed class of economies. $\mathcal{CC}$ satisfies OPIR and CONS on E and if φ is another solution that satisfies the two axioms on E, then $\varphi(\mathcal{E}) \subseteq \mathcal{CC}(\mathcal{E})$ for all $\mathcal{E} \in \mathsf{E}$.*

Proof: We know from Lemma 1 that the Coarse Core satisfies OPIR and CONS. Now, if φ satisfies OPIR and CONS, by Lemma 3 it must be that case that $\varphi(\mathcal{E}) \subseteq \mathcal{CC}(\mathcal{E})$ for all $\mathcal{E} \in \mathsf{E}$ since by Lemma 1, $\mathcal{CC}$ satisfies COCONS on E. □

We now state a characterization theorem:

Theorem 2 *A solution φ on a closed class E of economies satisfies CONS, COCONS, and OPR if and only if $\varphi = \mathcal{CC}$.*

Proof: By Lemma 1, the Coarse Core satisfies the three axioms. By Lemma 3, there cannot be two solutions that satisfy the three axioms. □

4.1.2 Independence of Axioms

All that remains to be shown is that the three axioms are independent. We do so by providing three examples. Consider the class of all economies.

1. The empty solution satisfies CONS and COCONS but does not satisfy OPR.

2. Consider the solution φ defined as:

$$\varphi(\mathcal{E}) = \begin{cases} \mathcal{IR}(\mathcal{E}) & \text{if } \mathcal{E} \text{ is a one-person economy} \\ \emptyset & \text{otherwise.} \end{cases}$$

The solution φ satisfies OPR since for any one-person economy $\mathcal{PO}$ and $\mathcal{IR}$ are the same. It satisfies CONS trivially but COCONS is not satisfied.

3. $\mathcal{PO}$ satisfies OPR and COCONS but by Theorem 2 it cannot satisfy CONS since $\mathcal{PO} \neq \mathcal{CC}$.

4.2 Axiomatization of the Coarse+ Core

In this subsection we provide an axiomatization of the Coarse+ Core, using the consistency principle. In order to apply this principle we need to define the appropriate notion of a reduced economy with respect to a coalition S and a status-quo $x \in \mathcal{A}(N)$. Here we follow the approach that in order to get the cooperation of a coalition outside the reduced economy, it is necessary to make a proposal that cannot be refused. Namely, the members of the outside coalition must be made better off at every state of the world.

The agents that compose this reduced economy are the members of S, and the F-allocations that each sub-coalition F of S can enforce are built as follows. If F is a strict subset of S, then in order to get the cooperation of any coalition $G \subseteq N \setminus S$ it is necessary to enforce with them a $S \cup G$-allocation that provides each member of G with a higher utility level at every state of the world than the utility of the status quo. A proposal like that has the property that no matter what the agents in G learn about the state of the world, they will still be willing to accept it rather than consuming their status quo bundle. Note that coalition F can perform this operation without the consent of the other members of the reduced economy. If $F = S$, they can get the cooperation of any coalition outside S in the same way just described, but in addition, they can enforce the status-quo x as well.

We begin by defining the set of bundles which dominate the status quo and the stand alone posibilities of an agent.

Definition 13 Let $i \in N$, and let $x_i \in \mathbb{B}_i$ be a bundle for i. The set of **non-refusable proposals** with respect to x_i is defined as:

$$NR_i^{x_i} = \left\{ z_i : \Omega \to \mathbb{R}^l \middle| u_i(z_i) > u_i(x_i) \right\}$$

That is, the set of non-refusable proposals for an agent i with respect to a bundle x_i is the set of all bundles which he strictly prefers in all states to the status quo. We now define the reduced economy which we will use to axiomatize the Coarse+ core.

Definition 14 Let $\mathcal{E} = \langle (X_i, \mathcal{F}_i, u_i, e_i)_{i \in N}, (\mathcal{A}(S))_{S \subseteq N} \rangle$ be an economy, let $x \in \mathcal{A}(N)$ be an allocation and let S be a coalition. The **reduced economy** with respect to x and S is defined as:

$$\mathcal{E}^{S,x} = \langle (X_i, \mathcal{F}_i, u_i, e_i)_{i \in S}, (\mathcal{A}^{S,x}(F))_{F \subseteq S} \rangle$$

where

$$\mathcal{A}^{S,x}(F) = \begin{cases} \bigcup_{G \subseteq N \setminus S} \{y_F : y \in \mathcal{A}(F \cup G) \text{ with } y_i \in NR_i^{x_i}, \forall i \in G\} \cup \{x_S\} & \text{if } F = S \\ \bigcup_{G \subseteq N \setminus S} \{y_F : y \in \mathcal{A}(F \cup G) \text{ with } y_i \in NR_i^{x_i}, \forall i \in G\} & \text{if } F \subsetneq S \end{cases}$$

The definitions of the consistency axiom and its converse remain unchanged, but it should be kept in mind that they are defined relative to the new reduced economies. The other axioms are not affected by the definition of the reduced economy. Furthermore, the very same arguments that appear after Definition 11 also show that among the classes of economies that are closed under this new reduction operation we can find:

1. The class of economies $\langle (X_i, \mathcal{F}_i, u_i, e_i)_{i\in N}, (\mathcal{A}(S))_{S\subseteq N}\rangle$ where for all $S \subseteq N$, and for all $i \in S$, if $y \in \mathcal{A}(S)$ then y_i is $\mathcal{F}_i$ measurable
2. The class of economies $\langle (X_i, \mathcal{F}_i, u_i, e_i)_{i\in N}, (\mathcal{A}(S))_{S\subseteq N}\rangle$ where for all $S \subseteq N$, and for all $i \in S$, if $y \in \mathcal{A}(S)$ then $y_i - e_i$ is $\cap_{k\in S}\mathcal{F}_i$ measurable
3. The class of economies where the S-allocations are incentive compatible.

Lemma 4 Let E be a closed class of economies. The Coarse+ Core satisfies OPR, PAR, CONS and COCONS on E.

Proof: We show that the Coarse+ Core satisfies each of the four axioms in order.

PAR An allocation that is not weakly efficient is not in the Coarse+ Core because the grand coalition improves upon it.

OPR For any one-person economy, the Coarse+ Core coincides with the set of individually rational allocations. If x is individually rational there is no state $\omega \in \Omega$ and allocation $y \in \mathcal{A}(\{i\})$ such that:

$$\mathrm{E}[u_i(y)|\mathcal{F}_i](\omega) > \mathrm{E}[u_i(x)|\mathcal{F}_i](\omega)$$

and it follows that x is in the Coarse+ Core. On the other hand, if x is not individually rational there exists a state ω and an allocation y such that:

$$\mathrm{E}[u_i(y)|\mathcal{F}_i](\omega) > \mathrm{E}[u_i(x)|\mathcal{F}_i](\omega)$$

but since the above conditional expectations are $\mathcal{F}_i$-measurable, this implies that $\{i\}$ Coarsely+ improves upon x.

CONS Let $x \in \mathcal{A}(N)$ be an allocation and assume that $x_S \notin \mathcal{C}^+(\mathcal{E}^{S,x})$. This means that there is a coalition $F \subseteq S$ that improves upon x_S. In other words there is an F-allocation $y_F \in \mathcal{A}^{S,x}(F)$ and a partition of F into two disjoint sets A and P such that:

1. $u_i(y_i) > u_i(x_i), \qquad \forall i \in P$
2. It is common knowledge among the members of A at some ω that:

$$\mathrm{E}[u_i(y_i)|\mathcal{F}_i] > \mathrm{E}[u_i(x_i)|\mathcal{F}_i], \qquad \forall i \in A$$

Since $y_F \in \mathcal{A}^{S,x}(F)$, there is a coalition $H \subseteq N \setminus S$ and allocation $y \in \mathcal{A}(F \cup H)$ such that:

1. $u_i(y_i) > u_i(x_i), \qquad \forall i \in H$
2. y_F is the projection of y on F.

Consider now $F \cup H$ and allocation $y \in \mathcal{A}(F \cup H)$. It follows that:

1. $u_i(y_i) > u_i(x_i), \qquad \forall i \in P \cup H$

2. It is common knowledge among the members of A at some ω that:

$$\mathrm{E}[u_i(y_i)|\mathcal{F}_i] > \mathrm{E}[u_i(x_i)|\mathcal{F}_i], \qquad \forall i \in A$$

But this means that $F \cup H$ Coarsely+ improves upon $x \in \mathcal{A}(N)$ and x is not in the Coarse+ core.

COCONS Assume that $x \notin \mathcal{C}^+(\mathcal{E})$. Then, there is a coalition S that Coarsely+ improves upon x. This means that there exists a state $\omega^* \in \Omega$, a partition $\{A, P\}$ of S and an S-allocation $y \in \mathcal{A}(S)$ such that

1. $u_i(y_i) > u_i(x_i), \qquad \forall i \in P$

2. It is common knowledge among the members of A at ω^* that: $(*)$

$$\mathrm{E}[u_i(y_i)|\mathcal{F}_i] > \mathrm{E}[u_i(x_i)|\mathcal{F}_i], \qquad \forall i \in A$$

If $A = N$ then x is not weakly efficient. If $A \neq N$ consider the reduced economy $\mathcal{E}^{A,x}$. By $(*)$, $y_A \in \mathcal{A}^{A,x}(A)$. Also by definition, $x_A \in \mathcal{A}^{A,x}(A)$. Consequently, by $(*)$ again, $x_A \notin \mathcal{C}^+(\mathcal{E}^{A,x})$. $\square$

Lemma 5 Let φ be a consistent and weakly efficient solution on E and let ψ be a converse consistent solution on the same class of economies. If $\varphi(\mathcal{E}) \subseteq \psi(\mathcal{E})$ for all one-person economies $\mathcal{E}$, then $\varphi(\mathcal{E}) \subseteq \psi(\mathcal{E}), \forall \mathcal{E} \in \mathsf{E}$.

Proof: The proof is by induction. The claim is trivially true for one-person economies. Suppose now that the statement of Lemma 5 holds for all k-person economies, $1 \leq k \leq N-1$. Let $x \in \varphi(\mathcal{E})$. Since φ satisfies PAR, x is weakly efficient. By CONS of $\varphi, x_F \in \varphi(\mathcal{E}^{F,x})$ for all $F \subseteq N, N \neq \emptyset, F \neq N$. By the induction hypothesis, $x_F \in \psi(\mathcal{E}^{F,x})$ for all $F \subseteq N, F \neq \emptyset, F \neq N$. Since ψ is converse consistent $x \in \psi(\mathcal{E})$. $\square$

As a corollary of the above result, we learn that the Coarse+ Core is the solution concept which is maximal with respect to set inclusion, among those that satisfy OPIR PAR and CONS.

Theorem 3 *Let E be a closed class of economies. $\mathcal{C}^+$ satisfies OPIR, PAR and CONS on E and if φ is another solution that satisfies the three axioms on E, then $\varphi(\mathcal{E}) \subseteq \mathcal{C}^+(\mathcal{E})$ for all $\mathcal{E} \in \mathsf{E}$.*

Proof: We know from Lemma 4 that the Coarse+ Core satisfies OPIR, PAR and CONS. Now, if φ satisfies OPIR, PAR and CONS, by Lemma 5 it must be that case that $\varphi(\mathcal{E}) \subseteq \mathcal{C}^+(\mathcal{E})$ for all $\mathcal{E} \in \mathsf{E}$ since by Lemma 4, $\mathcal{C}^+$ satisfies COCONS on E. $\square$

We now state a characterization theorem:

Theorem 4 *A solution φ on a closed class E of economies satisfies PAR, CONS, COCONS, and OPR if and only if $\varphi = \mathcal{C}^+$.*

Proof: By Lemma 4 the Coarse+ Core satisfies for the four axioms. By Lemma 5, there cannot be two solutions that satisfy the four axioms. $\square$

4.2.1 Independence of Axioms

All that remains to be shown is that the four axioms are independent. We do so by providing four examples. Consider the class of all economies.

1. $\mathcal{PO}$ satisfies OPR, PAR and COCONS but does not satisfy CONS by Theorem 3. (For a direct proof, consider a weakly efficient allocation in a three person economy which is (individualistically) improved upon by a two person coalition S and consider their corresponding reduced economy.)

2. The Empty Solution satisfies is CONS, COCONS and PAR but not OPR.

3. Consider the solution φ defined as:

$$\varphi(\mathcal{E}) = \begin{cases} \mathcal{IR}(\mathcal{E}) & \text{if } \mathcal{E} \text{ is a one-person economy} \\ \emptyset & \text{otherwise.} \end{cases}$$

   The solution φ is OPR, CONS and PAR but not COCONS.

4. The Individualistic Core $\mathcal{IC}$ satisfies OPR, CONS and COCONS but not PAR.

   Proof: We show that the $\mathcal{IC}$ satisfies OPR, CONS, and COCONS.

   OPR For any one-person economy, $\mathcal{IC}$ and $\mathcal{IR}$ coincide.

   CONS Suppose $x_s \notin \mathcal{IC}(\mathcal{E}^{S,x})$. This implies that there exists $F \subseteq S$ with $i \in F$ and a state ω such that F can (individualistically) improve upon x_F. That is, $\exists\, y \in \mathcal{A}^{S,x}(F)$ such that:

$$\mathrm{E}[u_i(y_i)|\mathcal{F}_i](\omega) > \mathrm{E}[u_i(x_i)|\mathcal{F}_i](\omega)$$

   and

$$y_k \in NR_k^{x_k} \qquad \forall k \in F \setminus \{i\}$$

   Since $y \in \mathcal{A}^{S,x}(F)$, there exists $G \subseteq N \setminus S$ and $y' \in \mathcal{A}(F \cup G)$ such that $y_k \in NR_k^{x_k}, \forall k \in G$ and $y'_F = y$. That is, y' improves upon x since:

$$y_k \in NR_k^{x_k} \qquad \forall k \in G \cup F \setminus \{i\}$$

   and

$$\mathrm{E}[u_i(y_i)|\mathcal{F}_i](\omega) > \mathrm{E}[u_i(x_i)|\mathcal{F}_i](\omega)$$

   COCONS The proof is similar to the proof that $\mathcal{C}^+$ satisfies COCONS and is left to the reader.

   Finally, we provide an example to show that the $\mathcal{IC}$ is not PAR. Consider the following economy with two agents, two commodities a and b, three states of the world, common prior $\mu = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, and constant across states utility functions for both agents given by $u(a,b) = \min\{a,b\}$.

		Endowment $(e_i)_N$			Allocation y		
Agent	$\mathcal{PF}_i$	ω_1	ω_2	ω_3	ω_1	ω_2	ω_3
1	$\{\{\omega_1\},\{\omega_2,\omega_3\}\}$	$(0,0)$	$(8,2)$	$(2,2)$	$(1,1)$	$(5,5)$	$(1,1)$
2	$\{\{\omega_1,\omega_2\},\{\omega_3\}\}$	$(2,2)$	$(2,8)$	$(0,0)$	$(1,1)$	$(5,5)$	$(1,1)$

Table 6

The endowment cannot be (individualistically) improved upon since there is no way to make either agent strictly better off in every state. However, the endowment is not weakly efficient since allocation y is strictly prefer at every state by both agents.

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