Unpublished Appendix to “Measuring School Segregation”*

David M. Frankel
Iowa State University

Oscar Volij
Iowa State University
and Ben Gurion University

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*Email addresses: dfrankel@econ.iastate.edu; ovolij@bgu.ac.il.

1 Introduction

This appendix proves the validity of Table 1 in Frankel and Volij [6], which is reproduced below. We also show which of the major segregation indices satisfy two decomposability properties.

		SYM	CI	SI	GDP	SDP	IND	N	CONT
1	Symmetric Atkinson: $A(X)$	√	√	√	×	√	√	√	√
2	Asymmetric Atkinson: $A_{\mathbf{w}}(X)$	×	√	√	×	√	√	√	√
3	Weighted Atkinson: $W(X)$	√	×	√	√	√	×	√	√
4	Lexicographic Atkinson: $\succcurlyeq_{\mathbf{w},\mathbf{w}'}$	×	√	√	×	√	√	√	×
5	Negative Atkinson: $-A(X)$	√	√	√	×	×	√	√	√
6	Mutual Information: $M(X)$	√	×	√	√	√	√	√	√
7	Asymmetric Mutual Information: $M_{(w_1,w_2)}(X)$	×	×	√	√	√	√	√	√
8	Negative Mutual Information: $-M(X)$	√	×	√	√	×	√	√	√
9	Scaled Mutual Information: $T(X)\,M(X)$	√	×	×	√	√	√	√	√
10	Weighted Dissimilarity: $D^W(X)$	√	2	√	×	√	×	√	√
11	Unweighted Dissimilarity: $D^U(X)$	√	√	√	×	√	×	√	√
12	Trivial Index	√	√	√	√	√	√	×	√
13	Gini Index: $G(X)$	√	2	√	×	√	×	√	√
14	Entropy Index: $H(X)$	√	×	√	×	√	√	√	√
15	Normalized Exposure: $NE(X)$	√	×	√	×	√	2	√	√
16	Clotfelter Index: $C(X)$	×	×	√	N/A	×	√	√	×
17	Card-Rothstein Index: $CR(X)$	×	×	√	N/A	×	×	√	√

Table 1: Which Indices Violate Which Axioms? A "√" means that the axiom is satisfied; an "×" indicates that it is not. A "2" means that the axiom is satisfied only in the case of two ethnic groups.

For definitions of the indices and the various properties, the reader is referred to Frankel and Volij [6]. We make our claims explicit by means of the following propositions. Our first proposition verifies the claims in the table.

Proposition 1 *The characterization in Table 1 is correct.*

Our second proposition characterizes which of the decomposability properties are satisfied by the major school segregation indices.

Proposition 2 *Mutual Information Index satisfies Strong School and Group Decomposability, while the Symmetric and Asymmetric Atkinson Indices, both Dissimilarity indices, and the Gini, Entropy, and Normalized Exposure indices violate them. In addition, the Clotfelter and Card-Rothstein indices violate Strong School Decomposability.*[1]

2 Proof of Proposition 1

The reader can check that, with the exception of Scaled Mutual Information, none of the indices in the table depend on the total district's population. Therefore, they satisfy SI. It is clear that Scaled Mutual Information does not satisfy SI, since for any district X such that $M(X) > 0$, and for any $\alpha > 1$ we have $T(\alpha X)M(\alpha X) = \alpha T(X)M(X) \neq T(X)M(X)$. It is also straightforward to check that all indices except for the trivial index satisfy Nontriviality.

It is immediate from the formulas for the various indices that all of them, except for the asymmetric version of the Atkinson and Mutual information orderings, as well as the lexicographic, the Clotfelter and the Card-Rothstein orderings, satisfy Symmetry.

All the orderings in the table with the exception of the lexicographic and the Clotfelter orderings, are represented by a continuous function. Consequently, as a trivial corollary of the following lemma we obtain that all of them, again with the exception of the above two, satisfy Continuity.

Lemma 1 *Any index that is a continuous function of the T_g^n's (the numbers of each group g in each school n) represents an ordering that satisfies the axiom of Continuity.*

[1] Strong Group Decomposability is not well defined for these indices, which apply only to a specific set of ethnic groups.

Proof. Let S be a segregation index. Assume that S is a continuous function of the T_g^n's. Fix a district Z with group set $\mathbf{G}$ and school set $\mathbf{N}$. The sets $(-\infty, S(Z)]$, and $[S(Z), \infty)$ are closed in $\Re$. Consequently, the intersections of $S^{-1}((-\infty, S(Z)])$ and $S^{-1}([S(Z), \infty))$ with $\mathcal{C}(\mathbf{G}, \mathbf{N})$ are closed in $\mathcal{C}(\mathbf{G}, \mathbf{N})$. (For continuous functions, their inverse image of closed sets are closed). But these are just the sets $\{X \in \mathcal{C}(\mathbf{G}, \mathbf{N}) : X \succcurlyeq Z\}$ and $\{X \in \mathcal{C}(\mathbf{G}, \mathbf{N}) : Z \succcurlyeq X\}$, respectively. Q.E.D.

The following example shows that the Clotfelter Index does not satisfy Continuity. Let $\kappa = .5$ and let $X(\varepsilon) = \langle (1-\varepsilon, 1), (0, 1) \rangle$ and $Z = \langle (1, 0), (1, 2) \rangle$, where in each school the first entry is the number of blacks. The set $\{X \in \mathcal{C}(\mathbf{G}, \mathbf{N}) : Z \succcurlyeq X\}$ is not closed since it contains $X(\varepsilon)$ for all $\varepsilon > 0$ but does not include $X(0)$, violating Continuity.

We now show that the lexicographic ordering does not satisfy Continuity. We will use the following lemma.

Lemma 2 *Let $\succcurlyeq$ be a segregation ordering that satisfies Continuity. Then for any districts $X, Y, Z \in \mathcal{C}$, the sets*

$$A = \{c \in [0,1] : cX \uplus (1-c)Y \succcurlyeq Z\} \text{ and } B = \{c \in [0,1] : Z \succcurlyeq cX \uplus (1-c)Y\}$$

are closed.

Proof. Let $\{c_k\}$ be a sequence of elements of A that converges to c. Then, $c_k X \uplus (1-c_k)Y$ is a sequence of districts in $\{X \in \mathcal{C}(\mathbf{G}, \mathbf{N}) : X \succcurlyeq Z\}$ that converges to $cX \uplus (1-c)Y$ (where $\mathbf{G}$ and $\mathbf{N}$ are the group and school sets of $cX \uplus (1-c)Y$). Since $\{X \in \mathcal{C}(\mathbf{G}, \mathbf{N}) : X \succcurlyeq Z\}$ is closed, $cX \uplus (1-c)Y \succcurlyeq Z$, which means that $c \in A$. A similar argument shows that B is closed. Q.E.D.

Let X and Y be two districts with different group distributions such that $A_{\mathbf{w}}(X) = A_{\mathbf{w}}(Y) < 1$ and $A_{\mathbf{w}'}(X) < A_{\mathbf{w}'}(Y)$. Let $c \in (0,1)$ and consider the district $cX \uplus (1-c)Y$. Let $\gamma_g = \frac{cT_g(X)}{cT_g(X)+(1-c)T_g(Y)}$ and $\eta_g = 1 - \gamma_g$. Note that a proportion $t_g^n(X)\gamma_g$ of group-g students of the district $cX \uplus (1-c)Y$ attend school $n \in \mathbf{N}(X)$. Likewise, a proportion $t_g^n(Y)\eta_g$ of group-g students of the district $cX \uplus (1-c)Y$ attend school $n \in \mathbf{N}(Y)$. Therefore, we can write

$$\begin{aligned}
1 - A_{\mathbf{w}}(cX \uplus (1-c)Y) &= \sum_{n\in\mathbf{N}(X)} \prod_{g\in\mathbf{G}} (t_g^n(X)\gamma_g)^{w_g} + \sum_{n\in\mathbf{N}(Y)} \prod_{g\in\mathbf{G}} (t_g^n(Y)\eta_g)^{w_g} \\
&= \sum_{n\in\mathbf{N}(X)} \prod_{g\in\mathbf{G}} \left(t_g^n(X)\right)^{w_g} \left(\gamma_g\right)^{w_g} + \sum_{n\in\mathbf{N}(Y)} \prod_{g\in\mathbf{G}} \left(t_g^n(X)\right)^{w_g} \left(\eta_g\right)^{w_g} \\
&= \left(\prod_{g\in\mathbf{G}} \left(\gamma_g\right)^{w_g}\right) \sum_{n\in\mathbf{N}(X)} \prod_{g\in\mathbf{G}} \left(t_g^n(X)\right)^{w_g} + \left(\prod_{g\in\mathbf{G}} \left(\eta_g\right)^{w_g}\right) \sum_{n\in\mathbf{N}(Y)} \prod_{g\in\mathbf{G}} \left(t_g^n(Y)\right)^{w_g} \\
&= (1 - A_{\mathbf{w}}(X)) \prod_{g\in\mathbf{G}} \left(\gamma_g\right)^{w_g} + (1 - A_{\mathbf{w}}(Y)) \prod_{g\in\mathbf{G}} \left(\eta_g\right)^{w_g}.
\end{aligned}$$

Since the group distributions of X and Y are not the same, there are groups $g, g' \in G$ with $\gamma_g \neq \gamma_{g'}$. (Otherwise, for all groups g, γ_g equals a constant λ, which implies $\frac{T_g(X)}{T_g(Y)} = \frac{\lambda(1-c)}{c(1-\lambda)}$. Hence, X and Y must have the same group distribution, a contradiction.) Therefore, the geometric average $\prod_{g\in\mathbf{G}} \left(\gamma_g\right)^{w_g}$ is strictly lower than the corresponding arithmetic average, and the same is true for $\prod_{g\in\mathbf{G}} \left(1-\gamma_g\right)^{w_g}$. As a result,

$$1 - A_{\mathbf{w}}(cX \uplus (1-c)Y) < (1 - A_{\mathbf{w}}(X)) \sum_{g\in\mathbf{G}} w_g\gamma_g + (1 - A_{\mathbf{w}}(Y)) \sum_{g\in\mathbf{G}} w_g\eta_g.$$

(By assumption, $A_{\mathbf{w}}(X)$ and $A_{\mathbf{w}}(Y)$ are strictly less than one.). Since $A_{\mathbf{w}}(X) = A_{\mathbf{w}}(Y)$, and since c was arbitrary chosen from $(0,1)$, we obtain that $A_{\mathbf{w}}(cX \uplus (1-c)Y) > A_{\mathbf{w}}(Y)$ for all $c \in (0,1)$. Consequently the set

$$\{c \in [0,1] : cX \biguplus (1-c)Y \succcurlyeq_{\mathbf{w},\mathbf{w}'} Y\}$$

equals $[0,1)$, which is not closed. By Lemma 2, $\succcurlyeq_{\mathbf{w},\mathbf{w}'}$ fails CONT.

For the rest of Table 1, we will proceed index by index.

2.1 Symmetric Atkinson

That the Atkinson indices satisfy the all the axioms except for GDP follows from Theorems 1 and 2.

GDP The symmetric Atkinson ordering does not satisfy GDP. Consider the districts $X = \langle(2,2),(0,2)\rangle$, and $X' = \langle(2,1,1),(0,1,1)\rangle$. The district X' is obtained from X by subdividing the second group into two equally sized, and equally distributed groups. It can be checked that $A(X) = 1 - 1/\sqrt{2}$ while $A(X') = 1 - 1/2^{2/3}$.

2.2 Asymmetric Atkinson

That the Atkinson indices satisfy the all the axioms except for GDP follows from Theorems 1 and 2. The previous example can be used to show that the Asymmetric Atkinson indices do not satisfy GDP.

2.3 Weighted Atkinson

GDP The weighted Atkinson ordering satisfies GDP. Let X be a school district and let X' be the school district that is obtained from X by subdividing ethnic group $g \in G(X)$ into two identically distributed ethnic groups g_1 and g_2. Note that since $T_g = T_{g_1} + T_{g_2}$, we have $P_g = P_{g_1} + P_{g_2}$. Also note that since both groups are identically distributed across schools, we have $t^n_{g_1} = t^n_{g_2} = t^n_g$ for all $n \in N$. Consequently

$$\left(t^n_{g_1}\right)^{P_{g_1}} \times \left(t^n_{g_2}\right)^{P_{g_2}} = \left(t^n_g\right)^{P_{g_1}} \times \left(t^n_g\right)^{P_{g_2}} = \left(t^n_g\right)^{P_g}$$

and as a result $W(X) = W(X')$.

SDP The weighted Atkinson ordering satisfies SDP. let X' be the result of splitting school n in district X into two schools, n_1 and n_2. Then $W(X') - W(X) = \prod\limits_{g\in\mathbf{G}} \left(t^n_g\right)^{P_g} - \prod\limits_{g\in\mathbf{G}} \left(t^{n_1}_g\right)^{P_g} - \prod\limits_{g\in\mathbf{G}} \left(t^{n_2}_g\right)^{P_g}$. Since $\sum_{g\in\mathbf{G}} P_g = 1$, the above products are homogeneous of degree one and concave functions of the vectors t^n, t^{n_1}, and t^{n_2}; since $t^n = t^{n_1} + t^{n_2}$, $W(X') - W(X) \geq 0$. If the school distributions in n_1 and n_2 are the same, then $t^{n_1}_g = \lambda t^n_g$ and $t^{n_2}_g = (1-\lambda)t^n_g$ for all g, where $\lambda = T^{n_1}/T^{n_2}$, so $W(X') - W(X) = 0$.

CI The weighted Atkinson index does not satisfy CI. Let $X = \langle(2,1),(1,2)\rangle$, and $X' = \langle(2,2),(1,4)\rangle$. It can be checked that $W(X) \neq W(X')$.

IND The weighted Atkinson index does not satisfy IND. To see this, consider the following districts: $X = \langle(1,0),(1,2)\rangle$, $Y = \langle(0,1),(2,1)\rangle$, and $Z = \langle(1,0)\rangle$. By symmetry we have $W(X) = W(Y)$. However, it can be checked that $W(X \uplus Z) < W(Y \uplus Z)$.

2.4 Lexicographic Atkinson

GDP The lexicographic Atkinson ordering does not satisfy GDP because the asymmetric Atkinson orderings don't.

SDP The lexicographic Atkinson ordering satisfies SDP because the Atkinson indices do.

CI The lexicographic Atkinson index satisfies CI because the Atkinson indices do.

IND The lexicographic Atkinson index satisfies IND because the Atkinson indices do.

2.5 Negative Atkinson

GDP The negative Atkinson ordering does not satisfy GDP because the symmetric Atkinson ordering doesn't.

SDP The negative Atkinson ordering does not satisfy SDP because the symmetric Atkinson ordering does.

CI The negative Atkinson index satisfies CI because the symmetric Atkinson ordering does.

IND The negative Atkinson index satisfies IND because the symmetric Atkinson ordering does.

2.6 Mutual Information

That the mutual information ordering satisfies all the axioms except for CI follows from Theorem 3. That it does not satisfy CI follows from Theorem 1 and the fact that the mutual information and the symmetric Atkinson are not the same ordering.

2.7 Asymmetric Mutual Information

GDP The Asymmetric Mutual Information ordering satisfies GDP. To see this, let X be a school district and let X' be the school district that is obtained by splitting group $g \in G \cap R_i$ into two identically distributed subgroups $g_1$and g_2 for some $i = 1, 2$. Then,

$$M_{w_1w_2}(X) - M_{w_1w_2}(X') = w_i[(k(P_g) - k(P_{g_1}) - k(P_{g_2}) - \sum_{n \in N} \pi^n \left(k(p_g^n) - k(p_{g_1}^n) - k(p_{g_2}^n)\right)]$$

where $k(q) = q \log_2(1/q)$. Letting $\alpha = P_{g_1}/P_g$ and noting that $\alpha = p_{g_1}^n / p_g^n$ for all $n \in N$, we have

$$\begin{aligned} M_{w_1w_2}(X) - M_{w_1w_2}(X') &= w_i \left[\begin{array}{c} \sum_{n \in N} \pi^n \left(\alpha \log_2 \frac{1}{\alpha} + (1-\alpha) \log_2 \frac{1}{1-\alpha}\right) - \\ \left(\alpha \log_2 \frac{1}{\alpha} + (1-\alpha) \log_2 \frac{1}{1-\alpha}\right) \end{array} \right] \\ &= 0 \end{aligned}$$

SDP The Asymmetric Mutual Information ordering satisfies SDP. To see this let X be a district and let n be a school of X. Let X' be the district that results from dividing n into two schools, n_1 and n_2. Since X and X' have the same group distribution,

$$\begin{aligned} M_{w_1w_2}(X') - M_{w_1w_2}(X) &= \sum_{i=1,2} (\pi^n H_i(p^n) - \pi^{n_1} H_i(p^{n_1}) - \pi^{n_2} H_i(p^{n_2})) \\ &= \pi^n \sum_{i=1,2} \left(H(p^n) - \frac{\pi^{n_1}}{\pi^n} H(p^{n_1}) - \frac{\pi^{n_2}}{\pi^n} H(p^{n_2}) \right) \end{aligned}$$

But for all g, $p_g^n = \frac{\pi^{n_1}}{\pi^n} p_g^{n_1} + \frac{\pi^{n_2}}{\pi^n} p_g^{n_2}$ so, recalling that $H_i((q_g)_{g \in \mathbf{G}}) = \sum_{g \in \mathbf{G} \cap \mathbf{R}_i} q_g \log_2(\frac{1}{q_g})$ is a concave function, $M_{w_1w_2}(X') - M_{w_1w_2}(X) \geq 0$, with strict inequality only if schools n_1 and n_2 have different group distributions.

CI The Asymmetric Mutual Information does not satisfy CI because the Mutual Information ordering doesn't.

IND The Asymmetric Mutual Information ordering satisfies IND. let X and Y have the same size and group distribution, with groups $1, ..., K$, and let Z be another district.

Then $X \uplus Z$ and $Y \uplus Z$ have the same group distribution, which we denote by P. Also, for each school n denote its group distribution by p^n. Then,

$$
\begin{aligned}
M_{w_1w_2}(X \uplus Z) \quad &\succcurlyeq \quad M_{w_1w_2}(Y \uplus Z) \\
&\iff \sum_{i=1,2} w_i \left(H_i(P) - \sum_{n \in \mathbf{N}(X)} \frac{T^n}{T(X \uplus Z)} H_i(p^n) - \sum_{n \in \mathbf{N}(Z)} \frac{T^n}{T(X \uplus Z)} H_i(p^n) \right) \\
&\geq \sum_{i=1,2} w_i \left(H_i(P) - \sum_{n \in \mathbf{N}(X)} \frac{T^n}{T(Y \uplus Z)} H_i(p^n) - \sum_{n \in \mathbf{N}(Z)} \frac{T^n}{T(Y \uplus Z)} H_i(p^n) \right) \\
&\iff \sum_{i=1,2} w_i \sum_{n \in \mathbf{N}(X)} T^n H_i(p^n) \leq \sum_{i=1,2} w_i \sum_{n \in \mathbf{N}(X)} T^n H_i(p^n) \\
&\iff M_{w_1w_2}(X) \succcurlyeq M_{w_1w_2}(Y).
\end{aligned}
$$

2.8 Negative Mutual Information

GDP The Negative Mutual Information ordering satisfies GDP because the Mutual Information ordering does.

SDP The Negative Mutual Information ordering does not satisfy SDP because the Mutual Information ordering does.

CI The Negative Mutual Information ordering does not satisfy CI because the Mutual Information ordering doesn't.

IND The Negative Mutual Information ordering satisfies IND because the Mutual Information ordering does.

2.9 Scaled Mutual Information

GDP The Scaled Mutual Information ordering satisfies GDP because the Mutual Information ordering does.

SDP The Scaled Mutual Information ordering satisfies SDP because the Mutual Information ordering does.

CI The Scaled Mutual Information ordering does not satisfy CI because it does not satisfy SI.

IND The Scaled Mutual Information ordering satisfies IND because the Mutual Information ordering does.

2.10 Weighted Dissimilarity

GDP To see that the Weighted Dissimilarity ordering does not satisfy GDP, consider the district $X = \langle (0,10), (10,0) \rangle$ and the district $Y = \langle (0,0,5,5), (5,5,0,0) \rangle$ that is obtained after we split evenly each of X's groups. It can be checked that $D^W(X) \neq D^W(Y)$.

SDP The Weighted Dissimilarity ordering satisfies SDP. To see this, let X be a district and let X' be the district that results from X if school $n \in X$ is divided into two schools, n_1 and n_2, and let $\alpha = T^{n_1}/T^n$. Then,

$$\begin{aligned} D^W(X') - D^W(X) &= \frac{1}{2I} \sum_{g \in \mathbf{G}(X)} \left(\pi^{n_1} \left| p_g^{n_1} - P_g \right| + \pi^{n_2} \left| p_g^{n_2} - P_g \right| - \left| p_g^n - P_g \right| \right) \\ &= \frac{1}{2I} \sum_{g \in \mathbf{G}(X)} \left(\alpha \left| p_g^{n_1} - P_g \right| + (1-\alpha) \left| p_g^{n_2} - P_g \right| - \left| p_g^n - P_g \right| \right) \end{aligned}$$

Since $p_g^n = \alpha p_g^{n_1} + (1-\alpha) p_g^{n_2}$ and absolute value is a convex function, $D^W(X') - D^W(X) \geq 0$. Moreover, if n_1 and n_2 have the same group distributions, $D^W(X') - D^W(X) = 0$.

CI The Weighted Dissimilarity ordering satisfies CI for two groups because it coincides with the Unweighted Dissimilarity ordering. To see that it does not satisfy CI in general, consider the following districts: $X = \langle (9,5,1), (1,5,9) \rangle$ and $Y = \langle (9,5,10), (1,5,90) \rangle$. It can be checked that $D^W(X) \neq D^W(Y)$.

IND To see that D^W violates IND, consider the following districts: $X = \langle (9,5), (1,5) \rangle$, $Y = \langle (5,9), (5,1) \rangle$ and $Z = \langle (6,4) \rangle$. Districts X and Y have the same population and group distribution. By symmetry of D^W, $D^W(X) = D^W(Y)$. However, it can be checked that $D^W(X \uplus Z) \neq D^W(Y \uplus Z)$.

2.11 Unweighted Dissimilarity

GDP To see that the Unweighted Dissimilarity ordering does not satisfy GDP, consider the district $X = \langle (0,10), (10,0) \rangle$ and the district $Y = \langle (0,0,5,5), (5,5,0,0) \rangle$ that is obtained after we split evenly each of X's groups. It can be checked that $D^U(X) \neq D^U(Y)$.

SDP The Unweighted Dissimilarity ordering satisfies SDP. To see this, let X be a district and let X' be the district that results from X if school $n \in X$ is divided into two schools, n_1 and n_2, and let $\alpha = T^{n_1}/T^n$. Then,

$$\begin{aligned}
D^U(X') - D^U(X) &= \frac{1}{2(K-1)} \sum_{g \in \mathbf{G}(X)} \left(\left| t_g^{n_1} - \sum_{g' \in \mathbf{G}} \frac{1}{K} t_{g'}^{n_1} \right| + \left| t_g^{n_2} - \sum_{g' \in \mathbf{G}} \frac{1}{K} t_{g'}^{n_2} \right| - \left| t_g^{n} - \sum_{g' \in \mathbf{G}} \frac{1}{K} t_{g'}^{n} \right| \right) \\
&\geq \frac{1}{2(K-1)} \sum_{g \in \mathbf{G}(X)} \left(\left| t_g^{n_1} + t_g^{n_2} - \sum_{g' \in \mathbf{G}} \frac{1}{K} \left(t_{g'}^{n_1} + t_{g'}^{n_2} \right) \right| - \left| t_g^{n} - \sum_{g' \in \mathbf{G}} \frac{1}{K} t_{g'}^{n} \right| \right) \\
&= \frac{1}{2(K-1)} \sum_{g \in \mathbf{G}(X)} \left(\left| t_g^{n} - \sum_{g' \in \mathbf{G}} \frac{1}{K} t_{g'}^{n} \right| - \left| t_g^{n} - \sum_{g' \in \mathbf{G}} \frac{1}{K} t_{g'}^{n} \right| \right) = 0.
\end{aligned}$$

Moreover, if n_1 and n_2 have the same group distributions, $t_g^{n_1} = t_g^{n_2}$ for all g, and consequently, $D^U(X') - D^U(X) = 0$.

CI The Unweighted Dissimilarity ordering satisfies CI since the corresponding index depends only on the t_g^ns.

IND To see that D^U violates IND, consider the following districts: $X = \langle (9,5), (1,5) \rangle$, $Y = \langle (5,9), (5,1) \rangle$ and $Z = \langle (6,4) \rangle$. Districts X and Y have the same population and group distribution. By symmetry of D^U, $D^U(X) = D^U(Y)$. However, it can be checked that $D^U(X \uplus Z) \neq D^U(Y \uplus Z)$.

2.12 Trivial index

It is obvious that this index satisfies all the axioms except for non-triviality.

2.13 Gini index

GDP To see that the Gini ordering does not satisfy GDP, consider the district $X = \langle (0,10), (10,0) \rangle$ and the district $Y = \langle (0,0,5,5), (5,5,0,0) \rangle$ that is obtained after we split evenly each of X's groups. It can be checked that $G(X) \neq G(Y)$.

SDP The Gini ordering satisfies SDP. Let X be a district, let X' be the district that results from X if school $n \in X$ is divided into two schools, n_1 and n_2, and let $\alpha = T^{n_1}/T^n$. We must show that $G(X') \geq G(X)$, with equality if the two schools have the same group distribution. But

$$\begin{aligned} G(X') - G(X) &= \frac{1}{I} \sum_{g=1}^{G} \frac{T^{n_1} T^{n_2}}{TT} \left| \frac{T_g^{n_1}}{T^{n_1}} - \frac{T_g^{n_2}}{T^{n_2}} \right| \\ &+ \frac{1}{I} \sum_{g=1}^{G} \sum_{\substack{m=1,\ldots,N \\ m \neq n}} \left(\begin{array}{c} \frac{T^m T^{n_1}}{TT} \left| \frac{T_g^m}{T^m} - \frac{T_g^{n1}}{T^{n_1}} \right| + \frac{T^m T^{n_2}}{TT} \left| \frac{T_g^m}{T^m} - \frac{T_g^{n2}}{T^{n2}} \right| \\ - \frac{T^m T^n}{TT} \left| \frac{T_g^m}{T^m} - \frac{T_g^n}{T^n} \right| \end{array} \right) \end{aligned}$$

The first sum is nonnegative. The summand in the second line equals

$$\frac{T^m}{TT} \left(\left| T^{n_1} \frac{T_g^m}{T^m} - T_g^{n_1} \right| + \left| T^{n_2} \frac{T_g^m}{T^m} - T_g^{n_2} \right| - \left| T^n \frac{T_g^m}{T^m} - T_g^n \right| \right)$$

The arguments of the first two absolute value functions sum to the argument of the third absolute value function. However, absolute value is a convex function. Hence, the summand is nonnegative for all g. Moreover, if the two schools have the same group distributions, then the arguments of the three absolute value functions are proportional to each other and thus all of the same sign. So the summand is zero.

CI In the case of two groups, the Gini Index can be written as $G(X) = \frac{1}{2} \sum_{n \in N(X)} \sum_{m \in N(X)} |t_1^n t_2^m - t_1^m t_2^n|$, so it clearly satisfies CI. To see that it does not satisfy CI in general, consider the following districts: $X = \langle (9,5,1), (1,5,9) \rangle$ and $Y = \langle (9,5,10), (1,5,90) \rangle$. It can be checked that $G(X) \neq G(Y)$.

IND To see that G violates IND, consider the following districts: $X = \langle (9,5), (1,5) \rangle$, $Y = \langle (5,9), (5,1) \rangle$ and $Z = \langle (6,4) \rangle$. Districts X and Y have the same population and group distribution. By symmetry of G, $G(X) = G(Y)$. However, it can be checked that $G(X \uplus Z) \neq G(Y \uplus Z)$.

2.14 Entropy

GDP The Entropy ordering does not satisfy GDP. To see this consider the district $X = \langle (0,10), (10,0) \rangle$ and the district $Y = \langle (0,0,5,5), (5,5,0,0) \rangle$ that is obtained after we split evenly each of X's groups. It can be checked that $H(X) \neq H(Y)$.

SDP The Entropy ordering satisfies SDP because the Mutual Information ordering does.

CI The Entropy ordering does not satisfy CI. Consider the following districts: $X = \langle (2,1), (1,2) \rangle$ and $Y = \langle (2,2), (1,4) \rangle$. It can be checked that $H(X) \neq H(Y)$.

IND The Entropy ordering satisfies IND. To see this, note that if X, Y are two districts with equal populations and equal group distributions, then (a) $E(X) = E(Y)$ and (b) $X \uplus Z$ and $Y \uplus Z$ have equal group distributions for all Z. Fact (b) implies that $E(X \uplus Z) = E(Y \uplus Z)$. Accordingly, $H(X) - H(Y) = \frac{M(X)-M(Y)}{E(Y)}$ and $H(X \uplus Z) - H(Y \uplus Z) = \frac{M(X \uplus Z)-M(Y \uplus Z)}{E(X \uplus Y)}$. Hence, H satisfies IND as M does.

2.15 Normalized Exposure

GDP The Normalized Exposure ordering does not satisfy GDP. To see this consider the district $X = \langle (0,10), (10,0) \rangle$ and the district $Y = \langle (0,0,5,5), (5,5,0,0) \rangle$ that is obtained after we split evenly each of X's groups. It can be checked that $NE(X) \neq NE(Y)$.

SDP The Normalized Exposure ordering satisfies SDP. Let X be a district, let X' be the district that results from X if school $n \in X$ is divided into two schools, n_1 and n_2.

Then

$$NE(X')-NE(X)=\sum_{g=1}^{G}\frac{1}{1-P_g}\left[\pi^{n_1}(p_g^{n_1}-P_g)^2+\pi^{n_2}(p_g^{n_2}-P_g)^2-\pi^{n}(p_g^{n}-P_g)^2\right].$$

Let $\alpha=\pi^{n_1}/\pi^n$. The expression between brackets is nonnegative iff

$$\alpha x^2+(1-\alpha)\,y^2\geq z \tag{1}$$

where $x=p_g^{n_1}-P_g$, $y=p_g^{n_2}-P_g$, and $z=p_g^{n}-P_g$. Since $\alpha x+(1-\alpha)\,y=z$, (1) holds iff

$$\begin{aligned}\alpha x^2+(1-\alpha)y^2 &\geq (\alpha x+(1-\alpha)\,y)^2\\ &= \alpha^2x^2+(1-\alpha)^2y^2+2\alpha\,(1-\alpha)\,xy\end{aligned}$$

which holds iff

$$(1-\alpha)\alpha x^2+\alpha(1-\alpha)y^2\geq 2\alpha(1-\alpha)xy$$

which is true since $(x-y)^2\geq 0$. Therefore, $NE(X')-NE(X)\geq 0$. If schools n_1 and n_2 have the same group distribution, then $p_g^{n_1}=p_g^{n_2}=p_g^{n}$ so $NE(X')-NE(X)=0$.

CI The Normalized Exposure ordering does not satisfy CI. Consider the following districts: $X=\langle(2,1)\,,(1,2)\rangle$ and $Y=\langle(2,2)\,,(1,4)\rangle$. It can be checked that $NE(X)\neq NE(Y)$.

IND To see why IND is satisfied with two groups, let $X,Y\in C$ be two districts with two groups, equal total size, and equal group distributions. Then for all districts Z with the same set of groups, we must show that $NE(X)\geq NE(Y)$ if and only if $NE(X\uplus Z)\geq NE(Y\uplus Z)$. Since $T(X)=T(Y)$ and $T_g(X)=T_g(Y)$ for all g,

$$NE(X)\geq NE(Y)\Longleftrightarrow\sum_{g\in\mathbf{G}}\sum_{n\in\mathbf{N}(X)}T^n\frac{\left(\frac{T_g^n}{T^n}-\frac{T_g(X)}{T(X)}\right)^2}{T(X)-T_g(X)}\geq\sum_{g\in\mathbf{G}}\sum_{m\in\mathbf{N}(Y)}T^m\frac{\left(\frac{T_g^m}{T^m}-\frac{T_g(Y)}{T(Y)}\right)^2}{T(Y)-T_g(Y)}$$

but $T_2^n=T^n-T_1^n$ and $T_2(X)=T(X)-T_1(X)$, so

$$\left(\frac{T_2^n}{T^n}-\frac{T_2(X)}{T(X)}\right)^2=\left(\frac{T_1^n}{T^n}-\frac{T_1(X)}{T(X)}\right)^2$$

and similarly,

$$\left(\frac{T_2^m}{T^m}-\frac{T_2(Y)}{T(Y)}\right)^2=\left(\frac{T_1^m}{T^m}-\frac{T_1(Y)}{T(Y)}\right)^2$$

so that

$$\begin{aligned}
\mathit{NE}(X)\geq \mathit{NE}(Y) &\Longleftrightarrow \sum_{n\in\mathbf{N}(X)} T^n\left(\frac{T_1^n}{T^n}-\frac{T_1(X)}{T(X)}\right)^2 \geq \sum_{m\in\mathbf{N}(Y)} T^m\left(\frac{T_1^m}{T^m}-\frac{T_1(X)}{T(X)}\right)^2\\
&\Longleftrightarrow \sum_{n\in\mathbf{N}(X)} T^n\left(\left(\frac{T_1^n}{T^n}\right)^2-2\frac{T_1^n}{T^n}\frac{T_1(X)}{T(X)}+\left(\frac{T_1(X)}{T(X)}\right)^2\right)\\
&\geq \sum_{m\in\mathbf{N}(Y)} T^m\left(\left(\frac{T_1^m}{T^m}\right)^2-2\frac{T_1^m}{T^m}\frac{T_1(X)}{T(X)}+\left(\frac{T_1(X)}{T(X)}\right)^2\right)\\
&\Longleftrightarrow \sum_{n\in\mathbf{N}(X)}\frac{(T_1^n)^2}{T^n}\geq\sum_{m\in\mathbf{N}(Y)}\frac{(T_1^m)^2}{T^m}
\end{aligned}$$

(In the first line we have eliminated the common factor $\frac{1}{T_1(X)}+\frac{1}{T_2(X)}$ and used the fact that $T_1(X)=T_1(Y)$ and $T(X)=T(Y)$.) A similar argument shows that

$$\begin{aligned}
&\mathit{NE}(X\uplus Z)\geq \mathit{NE}(Y\uplus Z)\\
&\qquad\Longleftrightarrow \sum_{n\in\mathbf{N}(X)} T^n\left(\frac{T_1^n}{T^n}-\frac{T_1(Z\uplus X)}{T(Z\uplus X)}\right)^2 \geq \sum_{m\in\mathbf{N}(Y)} T^m\left(\frac{T_1^m}{T^m}-\frac{T_1(Z\uplus X)}{T(Z\uplus X)}\right)^2\\
&\qquad\Longleftrightarrow \sum_{n\in\mathbf{N}(X)}\frac{(T_1^n)^2}{T^n}\geq\sum_{m\in\mathbf{N}(Y)}\frac{(T_1^m)^2}{T^m}
\end{aligned}$$

so *NE* satisfies IND in the case of two groups. IND is violated when there are more than two groups. A simple 3-group example suffices to show this. Let $X=\{(0,2,3),(6,4,3)\}$, $Y=\{(3,2,0),(3,4,6)\}$, and $Z=\{(0,10,100)\}$. Then $\mathit{NE}(X)=\mathit{NE}(Y)$ since *NE* satisfies SYM, but one can verify that $\mathit{NE}(X\uplus Z)\neq\mathit{NE}(Y\uplus Z)$.

2.16 Clotfelter

GDP It is not clear how to generalize this ordering to districts with more than two groups.

SDP The Clotfelter ordering does not satisfy SDP. To see this, suppose the one-school district X contains the school n in which the proportion minority is exactly κ. Assume there is more than one minority student in n. Suppose a new school is built and a single minority student from n is moved to that school. The effect of this is to remove $T_2^n - 1$ minority students from the sum in the definition of the index. Hence, the index falls, violating SDP.

CI The Clotfelter ordering does not satisfy CI. Consider the following districts: $X = \langle (2,1), (1,2) \rangle$ and $Y = \langle (2,2), (1,4) \rangle$. It can be checked that $C(X) \neq C(Y)$.

IND The Clotfelter ordering satisfies IND. To see this, let $X, Y \in \mathcal{C}$ be two districts with equal populations and equal group distributions. It is enough to show that for any district Z that contains a single school, $C(X) \geq C(Y)$ if and only if $C(X \uplus Z) \geq C(Y \uplus Z)$. Since $T_2(X) = T_2(Y)$,

$$\begin{aligned} C(X) \geq C(Y) &\iff \frac{1}{T_2(X)} \sum_{n \in \mathbf{N}(X): p_2^n \geq \kappa} T_2^n \geq \frac{1}{T_2(X)} \sum_{n \in \mathbf{N}(Y): p_2^n \geq \kappa} T_2^n \\ &\iff \frac{1}{T_2(X) + T_2(Z)} \sum_{n \in \mathbf{N}(X \uplus Z): p_2^n \geq \kappa} T_2^n \geq \frac{1}{T_2(X) + T_2(Z)} \sum_{n \in \mathbf{N}(Y \uplus Z): p_2^n \geq \kappa} T_2^n \\ &\iff C(X \uplus Z) \geq C(Y \uplus Z). \end{aligned}$$

2.17 Card-Rothstein

GDP It is not clear how to generalize this ordering to districts with more than three groups.

SDP The Card-Rothstein ordering does not satisfy SDP. To see this, consider the single school district $X = \langle (2,2,4) \rangle$, and the district $Y = \langle (1,0,4), (1,2,0) \rangle$ which is obtained after splitting that school in two. It can be checked that $CR(X) = 0$ while $CR(Y) = -1/15$.

CI The Card-Rothstein ordering does not satisfy CI. Consider the following districts: $X = \langle (9,5,1), (1,5,9) \rangle$ and $Y = \langle (9,5,10), (1,5,90) \rangle$. It can be checked that $CR(X) = 16/75$ while $CR(Y) = 7/48$.

IND The Card-Rothstein ordering does not satisfy IND. To see this, let $X = \langle (2,4,6), (6,4,2) \rangle$, $Y = \langle (4,2,1), (4,6,7) \rangle$ and $Z = \langle (0,2,5) \rangle$. It can be checked that $CR(X) = 1/12 < CR(Y) = 10/119$ while $CR(X \uplus Z) = 3/20 > CR(Y \uplus Z) = 88/595$.

Q.E.D.

3 Proof of Proposition 2

To see that M satisfies SSD, let $X = X^1 \uplus \cdots \uplus X^K$ be a district composed of K subdistricts. By definition of M, $M(X) = H(P(X)) - \sum_{k=1}^{K} \sum_{n \in \mathbf{N}(X^k)} \pi^n H(p^n)$. Subtracting and adding $\sum_{k=1}^{K} \pi^k H(P(X^k))$ on the right hand side, we obtain

$$\begin{aligned} M(X) &= H(P(X)) - \sum_{k=1}^{K} \pi^k H(P(X^k)) + \sum_{k=1}^{K} \pi^k H(P(X^k)) - \sum_{k=1}^{K} \sum_{n \in \mathbf{N}(X^k)} \pi^n H(p^n) \\ &= H(P(X)) - \sum_{k=1}^{K} \pi^k H(P(X^k)) + \sum_{k=1}^{K} \pi^k \left(H(P(X^k)) - \sum_{n \in \mathbf{N}(X^k)} \pi^n H(p^n) \right) \\ &= M(c(X^1) \uplus \cdots \uplus c(X^K)) + \sum_{k=1}^{K} \pi^k M(X^k), \end{aligned}$$

so M satisfies SSD. That M satisfies SGD follows from symmetry of mutual information (Cover and Thomas [5, pp. 18 ff.]).

We now show that the other orderings violate SSD. Let S satisfy SSD. Let X and Y have the same size and ethnic distribution, and let Z be another district. Then $c(X) = c(Y)$ and $T(X)/T(X \uplus Z) = T(Y)/T(Y \uplus Z) = p$. Then, applying SSD, $S(X \uplus Z) \geq S(Y \uplus Z)$ if and only if

$$\begin{aligned} S(c(X) \uplus c(Z)) + pS(X) + (1-p)S(Z) &\geq S(c(Y) \uplus c(Z)) + pS(Y) + (1-p)S(Z) \\ &\Leftrightarrow S(X) \geq S(Y) \end{aligned}$$

Hence, S also satisfies IND. By Table 1, both Dissimilarity indices, and the Gini, Normalized Exposure, and Card-Rothstein indices violate SSD. As for the Atkinson, Entropy, and Clotfelter indices, consider the district $X \uplus Y$ where $X = \langle (\alpha, 0), (0, 1-\alpha) \rangle$ and $Y = \langle (1-\alpha, 0), (0, \alpha) \rangle$. All three indices take the value one on X, on Y, and on $X \uplus Y$. In addition, for some values of α, they all take strictly positive values on $c(X) \uplus c(Y)$. (For the Clotfelter index, it suffices for α to exceed the threshold κ. For the other two indices, it is enough that $\alpha \neq 1/2$.) Hence, they also violate SSD.

We now turn to SGD. Suppose S satisfies SGD and assigns the value zero to any district in which all schools are representative. Let $X \in \mathcal{C}$ be a district in which the set of ethnic groups is G. Let X' be the result of partitioning some ethnic group $g \in G$ into two ethnic groups, g_1 and g_2, such that both ethnic groups have the same distribution across schools: $\frac{T^n_{g_1}}{T_{g_1}} = \frac{T^n_{g_2}}{T_{g_2}}$ for all $n \in \mathbf{N}$. Let $\widehat{X}$ be the result of removing all groups except g_1 and g_2 from district X'. Then by SGD,

$$S(X') = S(X) + \frac{T_g}{T} S\left(\widehat{X}\right) = S(X)$$

Accordingly, the ordering represented by S satisfies GDP. Since the indices other than M violate GDP (and equal zero on districts in which all schools are representative), they violate SGD as well. Q.E.D.

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