Pairwise consensus and the Borda rule*

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Abstract We say that a preference profile exhibits pairwise consensus around some fixed preference relation, if whenever a preference relation is closer to it than another one, the Kemeny distance of the profile to the former is not greater than its distance to the latter. We show that if a preference profile exhibits pairwise consensus around a preference relation, then this preference relation coincides with the binary relation induced by the Borda count. We also show that no other scoring rule always selects the top ranked alternative of the preference relation around which there is consensus when such consensus exists.

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1 Introduction

We study a standard setting of social choice in which there is a set of social alternatives and a group of voters, each of whom has a preference relation over this set. A preference

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aggregation rule identifies a subset of social preferences for every preference profile. The class of scoring rules is a well-known special class of preference aggregation rules in which each voter assigns a fixed list of K scores to the set of K alternatives according to their positions in his preference relation and rank the alternatives according to the sum of their scores. The Borda rule is an instance of a scoring rule in which the successive scores are equidistant. Another preference aggregation rule is the Kemeny rule, which selects those preference relations that minimize the sum of their Kemeny distances to the preferences in the profile.[1]

A minimal requirement of preference aggregation rules is that they satisfy the unanimity axiom. This axiom dictates that whenever there is unanimity on a given preference relation, namely, when all individuals share the same preference relation, the social preference must be this common preference relation. In this paper, we propose to strengthen the unanimity requirement by replacing the concept of unanimity on a preference relation by that of consensus around a preference relation. Roughly speaking, a preference profile exhibits pairwise consensus around some preference relation $\succ_0$ if whenever a preference relation is closer to $\succ_0$ than another one, the more in agreement are the preferences in the profile with the former than with the latter. Here, the level of agreement of a preference profile to a given preference relation is measured by the negative of the Kemeny distance. Our strengthened requirement, which we call the consensus property, dictates that whenever a preference profile exhibits consensus around some preference relation, the preference aggregation rule identifies precisely this preference relation as the social preference.

There are several preference aggregation rules that have the consensus property. In particular, the Kemeny rule satisfies it. This paper shows, however, that if we restrict attention to scoring rules, the only one that satisfies the pairwise consensus property is the Borda rule.

Although no scoring rule other than Borda satisfies the pairwise consensus property, one may suspect that the requirement that a preference profile exhibit consensus around some preference relation is strong enough to imply that some other scoring rules agree with Borda on their highest ranked alternatives. We show, however, that this is not

[1] See Kemeny and Snell (1962).

the case. Specifically, we show that for any scoring rule other than Borda, there is a preference profile exhibiting consensus around some preference relation whose top-ranked alternative differs from the scoring rule's top-ranked alternative.

One may wonder whether other scoring rules can be characterized by means of similar notions of consensus properties that differ only in the metric used to define them. We approach this question by showing that when the metric gives sufficiently higher weight to differences in the top-ranked alternatives, then consensus around a preference relation implies that this preference relation is consistent with the plurality rule, although it may not necessarily coincide with it. A similar observation holds for the inverse plurality rule.

Chebotarev and Shamis (1998) survey several existing characterizations of the Borda rule. In particular, we can mention Young (1974) and Nitzan and Rubinstein (1981). It is well known that for some preference profiles, the Kemeny and Borda rules disagree.[2] There are several articles that compare the Kemeny and Borda rules in terms of the properties they satisfy or fail to satisfy. Can and Storcken (2013), for instance, propose the axiom of *update monotonicity* which says that when a voter's preference is updated towards the current social preference, this social preference remains the social preference under the updated profile. They show that whereas the Kemeny rule satisfies update monotonicity, the Borda rule does not. One can also construct examples in which a profile exhibits consensus around some preference relation $\succ_0$ but this consensus is upset after one voter updates his preference relation towards $\succ_0$. Saari (2006), on the other hand, consider the *Neutral Condorcet Requirement*, which dictates that adding or removing preferences that constitute a Condorcet cycle to a given preference profile does not affect the social preferences. Saari (2006) shows that whereas the Borda rule fulfills this requirement, the Kemeny rule doesn't. Similarly, it can be shown that the addition of preferences that constitute a Condorcet cycle to a given profile which exhibits consensus may well upset this consensus. Nevertheless, we show that whenever there is consensus, the Kemeny and Borda rules coincide and furthermore, are consistent with majority rule.

The paper is organized as follows. Section 2 contains basic definitions. In Section

[2] See, for instance, Saari and Merlin (2000).

3 we define the concept of pairwise consensus and prove our main result.

2 Definitions

Let $A = \{a_1, \ldots, a_K\}$ be a set of $K \geq 2$ alternatives. Let $\mathcal{P}$ be the set of complete, transitive, and antisymmetric binary relations (also known as linear orders). We will refer to the elements of $\mathcal{P}$ as preference relations. For preference relation $\succ$, when we write $\succ = (a_1 a_2 ... a_K)$ we mean that a_1 is placed first in $\succ$, and so on. Let $\mathbb{N}$ be the set of non-negative integers, which represent the names of the potential voters. For any finite set $V \subseteq \mathbb{N}$ of voters, a preference profile is an assignment of a preference relation to each voter in V.

A *preference aggregation rule* is a function that assigns a nonempty subset of complete and transitive binary ralations on A to each preference profile. If $\succcurlyeq_\pi$ is a binary relation assigned by preference aggregation rule F to π, then the top-ranked alternatives according to $\succcurlyeq_\pi$ are said to be *selected* by F. A preference aggregation rule is *anonymous* if it is invariant to the names of the voters. In this paper, we consider only anonymous preference aggregation rules. Thus a preference profile can be summarized by a list $\pi = (\succ_1, \ldots, \succ_N)$ of individual preference relations where N is the number of voters. The class of preference profiles is, therefore, $\cup_{N \geq 1} \mathcal{P}^N$; namely, there is a fixed number K of alternatives and a variable number of voters.

A special class of anonymous preference aggregation rules consists of *scoring rules*. A scoring rule is characterized by K-tuple $S = (S_1, S_2, ..., S_K)$ of non-negative scores with $S_1 \geq S_2 \geq ... \geq S_K$ and $S_1 > S_K$. Given a preference profile π, each voter $i = 1, ..., N$ assigns S_k points to the alternative that is ranked k-th in his preference relation, for $k = 1, ..., K$. The scoring rule associated with the scores S, denoted by F_S, orders the alternatives according to their total scores. Many well-known preference aggregation rules are instances of scoring rules. For example, the *plurality rule* is the scoring rule associated with the scores $(1, 0, ..., 0)$. The *inverse plurality rule* is the scoring rule associated with scores $(1, ..., 1, 0)$. Lastly, the *Borda rule* is the scoring rule associated with the scores $(K-1, K-2, ..., 1, 0)$.

Let $d : \mathcal{P}^2 \to \mathbb{R}$ be the *inversion metric* on $\mathcal{P}$, which is defined as follows: $d(\succ, \succ')$ is the number of pairs of alternatives in A that are ranked differently by $\succ$ and $\succ'$.

Formally, the inversion metric is defined by

$$d(\succ, \succ') = \sum_{a \succ b} \mathbf{1}_{\succ'}(b, a)$$

where for any strict preference relation $\succ'$, $\mathbf{1}_{\succ'}(b, a) = 1$ if $b \succ' a$ and $\mathbf{1}_{\succ'}(b, a) = 0$ otherwise.

Example 1. Let the set of alternatives be $A = \{a, b, c\}$. The set $\mathcal{P}$ contain six preference relations given by: $\succ_1 = abc$, $\succ_2 = acb$, $\succ_3 = bac$, $\succ_4 = bca$, $\succ_5 = cab$, $\succ_6 = cba$. Consider the preference $\succ_1$. It can be checked that, according to the inversion metric, the distances of each preference in $\mathcal{P}$ to $\succ_1$ are given by

d	$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$	$\succ_5$	$\succ_6$
$\succ_1$	0	1	1	2	2	3

We can use the metric d to compare preference relations according to their "closeness" to some fixed preference relation. For any preference profile $\pi = (\succ_1, \dots, \succ_N)$ and any preference relation $\succ \in \mathcal{P}$, we denote by

$$d_\pi(\succ) = \sum_{n=1}^{N} d(\succ_n, \succ)$$

the Kemeny distance of π to $\succ$. It is the sum of the distances to $\succ$ of the voters' preferences.

3 Pairwise consensus

In this section we introduce the concept of pairwise consensus of preference profiles around a given preference relation.

Given a preference profile $\pi = (\succ_1, \dots, \succ_N)$ and two alternatives $a, b \in A$, we denote by

$$\mu_\pi(a \to b) = \sum_{n=1}^{N} \mathbf{1}_{\succ_n}(a, b).$$

the number of voters that prefer a to b. Note that the Borda count of alternative $a \in A$ for preference profile π is given by $BC_\pi(a) = \sum_{b \in A} \mu_\pi(a \to b)$. Also note that the Kemeny distance of π to $\succ$ can be written as

$$d_\pi(\succ) = \sum_{a \succ b} \mu_\pi(b \to a).$$

Indeed,

$$d_\pi(\succ) = \sum_{n=1}^{N} d(\succ_n, \succ) = \sum_{n=1}^{N} \sum_{a \succ b} \mathbf{1}_{\succ_n}(b, a) = \sum_{a \succ b} \sum_{n=1}^{N} \mathbf{1}_{\succ_n}(b, a) = \sum_{a \succ b} \mu_\pi(b, a).$$

Definition 1. A preference profile π exhibits pairwise consensus around preference relation $\succ_0$ if for all pairs of preference relations $\succ, \succ' \in \mathcal{P}$,

$$d(\succ, \succ_0) < d(\succ', \succ_0) \Longrightarrow d_\pi(\succ) \leq d_\pi(\succ')$$

with strict inequality if $\succ = \succ_0$.

This concept of consensus is similar to the concept of level-1 consensus introduced in Mahajne, Nitzan, and Volij (2015). In order to exhibit consensus around a preference relation $\succ_0$, both concepts require the fulfillment of a certain condition from a preference profile. This condition says that the closer to $\succ_0$ a preference relation is, the more similar (in some well-defined way) this preference relation should be to the preference profile. Whereas level-1 consensus measures similarity in terms of the number of voters that have the relevant preference relation, pairwise consensus measures it in terms of the Kemeny distance.

It is not the case that every preference profile exhibits pairwise consensus around some preference relation. However, when such consensus exists, it is around one and only one preference relation. Indeed, since $d(\succ_0, \succ_0) < d(\succ, \succ_0)$ for any two distinct preference relations, if preference profile π exhibits consensus around $\succ_0$, we have that

$$d_\pi(\succ_0) < d_\pi(\succ) \qquad \text{for all } \succ \neq \succ_0 . \tag{1}$$

This implies that π cannot exhibit consensus around any other preference relation other than $\succ_0$. Furthermore, equation 1 means that $\succ_0$ is the unique preference relation that minimizes the Kemeny distance to π.

Example 2. Continuing with Example 1, consider the following profile of four individuals: $\pi = (\succ_1, \succ_1, \succ_4, \succ_5)$. We obtain that the Kemeny distance of π to $\succ_i$, for $\succ_i \in \mathcal{P}$, is given by

$$d_\pi(\succ_i) = 2d(\succ_1, \succ_i) + d(\succ_4, \succ_i) + d(\succ_5, \succ_i).$$

These distances are summarized below:

	$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$	$\succ_5$	$\succ_6$
$d_\pi(\cdot)$	4	6	6	6	6	8

Comparing these distances with the ones calculated in Example 1, we can see that π exhibits pairwise consensus around $\succ_1$.

Given a preference profile π, we say that a beats b by majority voting, denoted $a \succ_M b$, if the number of individuals who prefer a to b is greater than the number of individuals who prefer b to a. Formally, the *binary relation induced by majority voting* is defined as follows:

$$a \succ_M b \Leftrightarrow \mu_\pi(a \to b) > \mu_\pi(b \to a).$$

We say a preference relation $\succ$ is *consistent with majority voting* if for all alternatives $a, b \in A$

$$a \succ_M b \Rightarrow a \succ b.$$

Finally, alternative a is a *Condordet winner* if $\mu_\pi(a \to b) \geq \mu_\pi(b \to a)$ for all $b \in A$. We can see that in Example 2, a is the unique Condorcet winner. Moreover, $\succ_1$ is consistent with majority voting. Indeed, we have both $a \succ_M b$ and $a \succ_1 b$ as well as $b \succ_M c$ and $b \succ_1 c$. However, $\succ_M$ does not coincide with $\succ_1$ since whereas $a \succ_1 c$, it is not the case that $a \succ_M c$.

The following claim presents some implications of consensus.

Observation 1. If $\pi = (\succ_1, \ldots, \succ_N)$ exhibits pairwise consensus around $\succ_0$, then

a. $\succ_0$ is consistent with majority voting.

b. The first ranked alternative in $\succ_0$ is the unique Condorcet winner.

c. If N is odd, $\succ_0$ coincides with $\succ_M$.

Proof. When $K = 2$, preference profile π exhibits pairwise consensus around $\succ_0$ if and only if $\succ_0 = \succ_M$ and the result is immediate. Therefore, let $K \geq 3$ and assume that π exhibits pairwise consensus around some preference relation $\succ_0$. Without loss of generality, assume that $\succ_0 = (a_1 \ldots a_K)$. In order to show that $\succ_0$ is consistent with majority voting, we will show that if $i < k$, then $\mu_\pi(a_i \to a_k) \geq \mu_\pi(a_k \to a_i)$. Fix $i < K$, and for $k = i+1, \ldots, K$, let $\succ^{i,k} = (a_1 a_2 \ldots a_{i-1} a_{i+1} \ldots a_k a_i a_{k+1} \ldots a_K)$ be the preference relation that is obtained from $\succ_0$ by moving alternative a_i from the ith rank to the kth rank. By construction, we have that $d(\succ_0, \succ^{i,k-1}) < d(\succ_0, \succ^{i,k})$ for $k = i+1, \ldots, K$. Since $\pi \in \mathcal{P}^n$ exhibits pairwise consensus around $\succ_0$, we must have that $d_\pi(\succ^{i,k-1}) \leq d_\pi(\succ^{i,k})$, with strict inequality when $k = i+1$ (because in that case $\succ^{i,k-1} = \succ_0$). But

$$\begin{aligned} d_\pi(\succ^{i,k}) &= \sum_{n=1}^{N} d(\succ_n, \succ^{i,k}) = \sum_{a \succ^{ik} b} \mu_\pi(b, a) \\ &= \sum_{1 < j \leq k} \mu_\pi(a_i \to a_j) + \sum_{\substack{h<j \\ (h,j) \neq (i,i+1), \ldots, (i,k)}} \mu_\pi(a_j \to a_h) \end{aligned}$$

Therefore,

$$0 \leq d_\pi(\succ^{i,k}) - d_\pi(\succ^{i,k-1}) = \mu_\pi(a_i \to a_k) - \mu_\pi(a_k \to a_i), \tag{2}$$

with strict inequality when $k = i+1$. Since this is true for all $1 \leq i < k \leq K$, it means that $\succ_0$ is consistent with majority voting. In particular, for $i = 1$ it means that the top-ranked alternative in $\succ_0$ is a Condorcet winner. In fact, it is the unique Condorcet winner since for any other alternative $a_k \neq a_1$ we have that $\mu_\pi(a_{k-1} \to a_k) > \mu_\pi(a_k \to a_{k-1})$, and therefore a_k cannot be a Condorcet winner. Finally, when N is odd, all the above inequalities become strict since $\mu_\pi(a_i \to a_k) \neq \mu_\pi(a_k \to a_i)$, and consequently $\succ_0 = \succ_M$. □

Although pairwise consensus around a preference relation implies that this preference relation is consistent with majority voting, it may well be the case that the majority tournament is a linear order even if there is no consensus. The following example illustrates this point.

Example 3. Consider the following preference profile: $\pi = (abc, abc, bac, bca, cab)$. We obtain that $a \succ_M b$, $b \succ_M c$, and $a \succ_M c$, which means that the binary relation induced by majority voting is a linear order and the alternative a is the unique Condorcet winner. Furthermore, the Kemeny order is abc. However, we have that $d_\pi(acb) =$

$8 > 7 = d_\pi(bca)$ even though $d(abc, acb) < d(abc, bca)$. That is, π does not exhibit consensus around abc. In fact, π does not exhibit consensus around any preference relation.

We now state our main result.

Theorem 1. If preference profile π exhibits pariwise consensus around preference relation $\succ_0$, then $\succ_0$ coincides with the order induced by the Borda count. That is, for all $a, b \in A$, $BC_\pi(a) > BC_\pi(b) \Leftrightarrow a \succ_0 b$.

Proof. Assume without loss of generality that $\pi \in \mathcal{P}^n$ exhibits pairwise consensus around $\succ_0 = (a_1 \dots, a_K)$. Fix two alternatives $a_j \neq a_i$ with $a_i \succ_0 a_j$. We need to show that $BC_\pi(a_i) > BC_\pi(a_j)$. Note that

$$BC(a_i) - BC(a_j) = \sum_{a_k \neq a_i} \mu_\pi(a_i \to a_k) - \sum_{a_k \neq a_j} \mu_\pi(a_j \to a_k)$$

$$= \sum_{a_k \neq a_i} \frac{\mu_\pi(a_i \to a_k) - (n - \mu_\pi(a_i \to a_k))}{2} - \sum_{a_k \neq a_j} \frac{\mu_\pi(a_j \to a_k) - (n - \mu_\pi(a_j \to a_k))}{2}$$

$$= \sum_{a_k \neq a_i} \frac{\mu_\pi(a_i \to a_k) - \mu_\pi(a_k \to a_i)}{2} - \sum_{a_k \neq a_j} \frac{\mu_\pi(a_j \to a_k) - \mu_\pi(a_k \to a_j)}{2}$$

Therefore, it is enough to show that

$$X := \sum_{a_k \neq a_i} [\mu_\pi(a_i \to a_k) - \mu_\pi(a_k \to a_i)] - \sum_{a_k \neq a_j} [\mu_\pi(a_j \to a_k) - \mu_\pi(a_k \to a_j)] > 0.$$

Note that

$$X = \sum_{i < k \leq j} \underbrace{[\mu_\pi(a_i \to a_k) - \mu_\pi(a_k \to a_i)]}_{\geq 0} - \sum_{i \leq k < j} \underbrace{[\mu_\pi(a_j \to a_k) - \mu_\pi(a_k \to a_j)]}_{\leq 0}$$

Since $\succ_0$ is consistent with majority voting, $X \geq 0$. Furthermore, in the proof of Observation 1 (see inequality (2)), it is shown that $\mu_\pi(a_i \to a_{i+1}) - \mu_\pi(a_{i+1} \to a_i) > 0$ and that $\mu_\pi(a_j \to a_{j-1}) - \mu_\pi(a_{j-1} \to a_j) < 0$. Therefore, we conclude that $X > 0$. $\square$

The above results suggest that the requirement that a preference profile exhibit consensus around some preference relation $\succ_0$ is rather strong, since it implies both that $\succ_0$ is a Kemeny preference and that it is represented by the Borda count. One may wonder whether it is so strong as to imply that other scoring rules also rank $\succ_0$'s top ranked alternative at the top of their ordering. Proposition 1 below shows that this is not the case.

Proposition 1. The Borda rule is the unique scoring rule that, whenever there is pairwise consensus around $\succ_0$, selects the top ranked alternative according to $\succ_0$.

Proof. Theorem 1 already shows that the Borda rule selects the top ranked alternative of the preference relation around which there is pairsiwe consensus. We now show that for any scoring rule F_S other than Borda, there is a preference profile that exhibits consensus around $\succ_0 = (a_1 \dots a_K)$ but F_S does not select a_1. Let F_S be a scoring rule different from Borda defined by the scores $S = (S_1, S_2, \dots, S_K)$. Without loss of generality assume that $1 = S_1 \geq \dots \geq S_K = 0$. The fact that F_S is not the Borda rule means that there is some index j such that $S_j + S_{K+1-j} \neq 1$. Asume first that $S_j + S_{K+1-j} > 1$ for some j, and consider the preference profile π in which n individuals have preference relation $\succ_0 = (a_1 \dots a_K)$ and $n-1$ individuals have the reverse preference relation $\succ_0^{-1} = (a_K \dots a_1)$. The other preferences are not present in the profile. Then,

$$\begin{aligned} d_\pi(\succ) &= nd(\succ_0, \succ) + (n-1)d(\succ_0^{-1}, \succ) \\ &= d(\succ_0, \succ) + (n-1)\big(d(\succ_0, \succ) + d(\succ_0^{-1}, \succ)\big) \end{aligned}$$

Note that for any preference relation $\succ$, we have that

$$d(\succ_0, \succ) + d(\succ_0^{-1}, \succ) = d(\succ_0, \succ_0^{-1}) \qquad \text{for all } \succ \in \mathcal{P}.$$

Therefore

$$d_\pi(\succ) = d(\succ_0, \succ) + (n-1)d(\succ_0, \succ_0^{-1})$$

which implies that π exhibits pairwise consensus around $\succ_0$. On the other hand, the scores awarded by π to the various alternatives are given by

$$\text{Score}(a_k) = nS_k + (n-1)S_{K+1-k} \qquad \text{for } k = 1, 2, \dots, K.$$

In particular, $\text{Score}(a_1) = n$. Therefore,

$$\begin{aligned}\text{Score}(a_1) < \text{Score}(a_j) &\Leftrightarrow n < nS_j + (n-1)S_{K+1-j} \\ &\Leftrightarrow S_{K+1-j} < n(S_j + S_{K+1-j} - 1) \\ &\Leftrightarrow \frac{S_{K+1-j}}{S_j + S_{K+1-j} - 1} < n.\end{aligned}$$

That is, for sufficiently large n, F_S does not select a_1.

Assume now that $S_k + S_{K+1-k} \leq 1$ for all k , with strict inequality for at least one k. Denote that index by j. Consider a preference profile in which one individual has preference relation $\succ_0 = (a_1 \dots a_K)$, n individuals have preference relation $\succ_j = (a_j, a_2 \dots a_{j-1} a_1 a_{j+1} \dots a_K)$, and n individuals have preference relation $\succ_j^{-1} = (a_K \dots a_{j+1} a_1 a_{j-1} \dots a_2 a_j)$. Preference relation $\succ_j$ is obtained from $\succ_0$ by swapping alternatives a_1 and a_j. Preference relation $\succ_j^{-1}$ is obtained by reversing $\succ_j$. Then, we have

$$\begin{aligned}d_\pi(\succ) &= d(\succ_0, \succ) + nd(\succ_j, \succ) + nd(\succ_j^{-1}, \succ) \\ &= d(\succ_0, \succ) + n\left(d(\succ_j, \succ) + d(\succ_j^{-1}, \succ)\right) \\ &= d(\succ_0, \succ) + nd(\succ_j, \succ_j^{-1})\end{aligned}$$

We can see that π exhibits consensus around $\succ_0$. The scores obtained by a_1 and a_j are given, respectively, by

$$\begin{aligned}\text{Score}(a_1) &= 1 + nS_j + nS_{K+1-j} \\ \text{Score}(a_j) &= S_j + n\end{aligned}$$

Therefore,

$$\begin{aligned}\text{Score}(a_1) < \text{Score}(a_j) &\Leftrightarrow 1 + nS_j + nS_{K+1-j} < S_j + n \\ &\Leftrightarrow 1 - S_j < n(1 - S_j - S_{K+1-j}) \\ &\Leftrightarrow \frac{1 - S_j}{1 - S_j - S_{K+1-j}} < n\end{aligned}$$

That is, for large enough n, F_S does not select a_1. □

3.1 Plurality

It would be interesting to see if other scoring rules can be characterized by means of some similar notions of the consensus property that differ only in the metric used to define them, which we proceed to do in the following.

Given a preference profile π, the *plurality count* of alternative a is the number $P_\pi(a)$ of voters that place a on top of their preference relation. Formally, $P_\pi(a) = |\{n : a = top(\succ_n)\}|$, where $top(\succ)$ denotes the top-ranked alternative according to $\succ$. We say that a preference relation $\succ$ is *consistent with the plurality rule* if for all alternatives $a, b \in A$,

$$P_\pi(a) > P_\pi(b) \Rightarrow a \succ b.$$

Consider the following distance on $\mathcal{P}^2$:

$$d^M(\succ, \succ') = \begin{cases} M + d(\succ, \succ') & \text{if } top(\succ) \neq top(\succ') \\ d(\succ, \succ') & \text{if } top(\succ) = top(\succ') \end{cases}$$

where d is the Kemeny distance and M is a positive integer. The associated distance of a preference profile to a given preference relation is given by

$$\begin{aligned} d^M_\pi(\succ) &= \sum_{n=1}^{N} d^M(\succ_n, \succ) \\ &= (N - P_\pi(top(\succ)))\, M + \sum_{n=1}^{N} d(\succ_n, \succ) \end{aligned}$$

With this modified metric, the definition of d^M-consensus is, *mutatis mutandis*, the one that appears in Definition 1.

Proposition 2. Let $M > N \max d(\succ, \succ')$. If π exhibits d^M-consensus around $\succ_0$, then $\succ_0$ is consistent with the plurality rule.

Proof. Assume without loss of generality that π exhibits d^M consensus around $\succ_0 = (a_1 \dots a_K)$. Let a_i and a_j be two alternatives such that $a_i \succ_0 a_j$ and let $\succ_i$ be the preference relation that is obtained from $\succ_0$ by promoting a_i to the top. Similarly, let $\succ_j$ be the preference relation that is obtained from $\succ_0$ by promoting a_j to the top. We have that $d^M(\succ_i, \succ_0) < d^M(\succ_j, \succ_0)$. Therefore, $d^M_\pi(\succ_i) \leq d^M_\pi(\succ_j)$. As a result,

$$\begin{aligned} d^M_\pi(\succ_i) \leq d^M_\pi(\succ_j) &\Leftrightarrow (N - P_\pi(top(\succ_i)))\, M + d_\pi(\succ_i) \leq (N - P_\pi(top(\succ_j)))\, M + d_\pi(\succ_j) \\ &\Leftrightarrow (P_\pi(a_j) - P_\pi(a_i))\, M \leq d_\pi(\succ_j) - d_\pi(\succ_i) \\ &\Leftrightarrow (P_\pi(a_j) - P_\pi(a_i)) \leq \frac{d_\pi(\succ_j) - d_\pi(\succ_i)}{M} < 1 \\ &\Rightarrow (P_\pi(a_j) - P_\pi(a_i)) \leq 0 \end{aligned}$$

Namely, alternative a_i is ranked at least as high as a_j by the plurality count. $\square$

Although a preference relation around which there is consensus is consistent with the plurality count, it may not coincide with the ranking induced by this count as the following example illustrates.

Example 4. Let $A = \{a, b, c\}$, and consider a preference profile π containing three copies of *abc*, two copies of *acb*, five copies of *bac*, and four copies of *cab*. It can be checked that the distances of the different preferences to $\succ_1$ and to π are given by

d	abc	acb	bac	bca	cab	cba
abc	0	1	$1+M$	$2+M$	$2+M$	$3+M$
π	$15+9M$	$17+9M$	$19+9M$	$25+9M$	$23+10M$	$27+10M$

and therefore, for $M \geq 2$, π exhibits d^M-consensus around $\succ_1$. It can also be seen that whereas $a \succ_1 b$ we have that $P_\pi(a) = P_\pi(b) = 5$. We conclude that the plurality rule does not necessarily select the top ranked alternative of the preference relation around which there is d^M-consensus.

Remark. If we used the pseudo-metric

$$\hat{d}(\succ, \succ') = \begin{cases} 1 & \text{if } top(\succ) \neq top(\succ') \\ 0 & \text{if } top(\succ) = top(\succ') \end{cases}$$

a similar argument to the above would show that if there is $\hat{d}$-consensus around $\succ_0$ then $\mu_\pi(a_j) - \mu_\pi(top(a_0)) < 0$, namely, the plurality rule uniquely selects the top rank alternative according to $\succ_0$.

Remark. A similar analysis shows that if we used the metric

$$d_M(\succ, \succ') = \begin{cases} M + d(\succ, \succ') & \text{if } bot(\succ) \neq bot(\succ') \\ d(\succ, \succ') & \text{if } bot(\succ) = bot(\succ') \end{cases}$$

where $bot(\succ)$ stands for the bottom ranked alternative in $\succ$, we would obtain that for $M > N \max d(\succ, \succ')$, if π exhibits d_M-consensus around $\succ_0$, then $\succ_0$ is consistent with the inverse plurality rule.

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