Static Games*

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Article Outline

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Glossary and Notation

Player A participant in a game.

Action set The set of actions that a player may choose.

Action profile A list of actions, one for each player.

Payoff The utility a player obtains from a given action profile.

1 Definition of the subject and its importance

Game theory concerns the interaction of decision makers. This interaction is modeled by means of games. There are various approaches to constructing games. One approach is to focus on the possible outcomes of the decision-makers' interaction by abstracting from the actions or decisions that may lead to these outcomes. The main tool used to implement this approach is the *cooperative game*. Another approach is to focus on the actions that the decision-makers can take, the main tool being the *non-cooperative game*. Within this approach, strategic interactions are modeled in two ways. One is by means of *dynamic*, or extensive form games, and the other is by means of *static*, or strategic games. Dynamic games stress the sequentiality of the various decisions that agents can make. An essential component of a dynamic game is the description of who moves first, who moves second, etc. Static games, on the other hand, abstract from the sequentiality of the possible moves, and model interactions as simultaneous decisions, where the decisions may well be complicated plans of actions that dictate different moves for different situations that may arise. All extensive form games can be modeled as static games, and all strategic form games can be modeled as extensive form games. But some situations may be more conveniently modeled as one or the other kind of game.

This chapter reviews the main ideas and results related to static games, as well as some interesting relationships that connect equilibrium concepts with the idea of rationality. The objective is to introduce the reader to the area of static games and to stimulate his interest for further knowledge of game theory in general. For a comprehensive exposition of some results not covered in this chapter, the reader is referred to the many excellent textbooks available on game theory. Fudenberg and Tirole (1991) Osborne and Rubinstein (1994), Osborne (2004), Binmore (2007) constitute only a partial list.

Although the definition of a static game is a very simple one, static games are a very flexible model which allows us to analyze many different situations. In particular, one can use them to analyze strategic interactions that involve either common interests or diametrically opposed interests. Similarly, one can also use static games to model situations where players have either symmetric or asymmetric information. The range of applications of static games is very wide and covers many disciplines, such as economics, political science, biology, philosophy, and computer science among others.

2 Introduction

In this section we introduce some examples that will be used later to motivate different concepts. We also introduce the definition of a static game.

The prisoner's dilemma involves a donor who is interested in donating some amount of money to two universities. The donor decides that the amount each university will receive depends on the content of the messages the presidents of the respective universities will send to him. Each university will send simultaneously one of two messages. One possible message is "Give him 2" and the other is "Give me 1." The donor will do exactly as told. For instance, if University I sends the message "Give me 1" and University II sends "Give him 2," the donor will donate \$3 to University I and \$0 to University II. This game can be described by means of the following matrix, where the entries represent the payoffs for University I and University II, respectively, that result from the corresponding action choices:

		University II	
		Give him 2	Give me 1
University I	Give him 2	2, 2	0, 3
	Give me 1	3, 0	1, 1

The battle of the sexes consists of two friends, She and He, who want to go out together, but have no means of communication. They have to decide, each one separately but both simultaneously, whether to go to a boxing match or to a ballet show. For both of them, the worst possible outcome would be to choose different events and not meet. But if they meet, he would rather meet her at the boxing match, while she would rather meet him at the ballet. The battle of the sexes can be described by the following matrix:

		She	
		Box	Ballet
He	Box	2, 1	0, 0
	Ballet	0, 0	1, 2

Again, the entries of this matrix represent the payoffs that he and she get, as a result of their corresponding choices.

Chicken models two drivers who approach each other on a narrow street. If none of them slows down they'll have an accident and their corresponding payoffs will be 0. But if at least one of them slows down, the accident is prevented. The problem is that both of them would like the other to slow down. If only one driver slows down, this driver gets a payoff of 2 and the other driver gets a payoff of 7. If both drivers slow down, then both drivers get a payoff of 6. This situation can be described by the following matrix.

		Driver 2	
		Slow Down	Speed up
Driver 1	Slow Down	6, 6	2, 7
	Speed up	7, 2	0, 0

Matching Pennies involves two friends, each of whom places a coin on a table. If both coins are placed heads up or tails up, then friend 1 gets one dollar from friend 2. If one coin is placed heads up and the other tails up, then friend 1 pays one dollar to friend 2. Matching pennies can be described by the following matrix, where the entries are the amounts of money that the friends get from each other.

		Friend 2	
		Heads	Tails
Friend 1	Heads	1, -1	-1, 1
	Tails	-1, 1	1, -1

The above examples of strategic interactions can be modeled as static games. A static game is a formalization of a strategic situation according to which players choose their actions separately and simultaneously, and as a result obtain certain payoffs. The interaction that a static game models need not require that players take their actions simultaneously. But the interaction is modeled by defining actions in such a way that lets us think of the players as acting simultaneously.

All of the above examples involve a set of players, and for each player there is a set of available actions and a function that associates a payoff level to each of the profiles of actions that may result from the players' choices. These are the three essential components of a static game, as formalized in the following definition.

Definition 1 A static game is a triple $\langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ where N is a finite set of players, and for each player $i \in N$, A_i is i's set of actions, and $u_i : \times_{k\in N} A_k \to \mathbb{R}$ is player i's utility function.

In the prisoner's dilemma the set of players is $N = \{\text{University I, University II}\}$; the sets of actions are $A_I = A_{II} = \{$*Give me 1, Give him 2*$\}$; the utility function of University I is $u_I($*Give me 1, Give me 1*$) = 1$, $u_I($*Give me 1, Give him 2*$) = 3$, $u_I($*Give him 2, Give me 1*$) = 0$, $u_I($*Give him 2, Give him 2*$) = 2$; and the utility function of University II is $u_{II}($*Give me 1, Give me 1*$) = 1$, $u_{II}($*Give me 1, Give him 2*$) = 0$, $u_{II}($*Give him 2,Give me 1*$) = 3$, $u_I($*Give him 1, Give him 1*$) = 1$.

In this chapter we sometimes refer to static games simply as games. For any game $\langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$, the set of action profiles $\times_{k\in N} A_k$ is denoted by A, and a typical action profile is denoted by $a = (a_i)_{i\in N} \in A$. If A is a finite set, then we say that the game is finite. Player i's utility function represents his preferences over the set of action profiles. For instance, for any two action profiles a and a' in A, $u_i(a) \geq u_i(a')$ means that player i prefers action profile a to action profile a'. Clearly, although player i has preferences over action profiles, he can only affect his own component, a_i, of the profile.

3 Nash equilibrium

One objective of game theory is to select, for each game, a set of action profiles that are interesting in some way. These action profiles may be interpreted as predictions of the theory, or prescriptions for the players to follow, or simply as equilibrium outcomes in the sense that if they occur, the players do not wish that they had acted differently. These action profiles are formally given by solution concepts, which are functions that associate each strategic game with the selected set of action profiles. The central solution concept in game theory is known as *Nash equilibrium*. The hypothesis behind this solution concept is that each player chooses his actions so as to maximize his utility, given the profile of actions chosen by the other players. To give a formal definition of the Nash equilibrium concept, we first introduce some useful notation. For each player $i \in N$, let $A_{-i} = \times_{k \in N \setminus \{i\}} A_k$ be the set of the other players' profiles of actions. Then we can write $A = A_i \times A_{-i}$, and each action profile can be written as $a = (a_i, a_{-i}) \in A_i \times A_{-i}$, thereby distinguishing player i's action from the other players' profile of actions.

Definition 2 The action profile $a^* = (a_i^*)_{i \in N} \in A$ in a game $\langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is a *Nash equilibrium* if for each player, $i \in N$, and every action $a_i \in A_i$ of player i, a^* is at least as good for player i as the action profile (a_i, a_{-i}^*). That is, if

$$u_i(a^*) \geq u_i(a_i, a_{-i}^*) \qquad \text{for all } a_i \in A_i \quad \text{and for all } i \in N.$$

It is a *strict Nash equilibrium* if the above inequality is strict for all alternative actions $a_i \in A_i \setminus \{a_i^*\}$.

3.1 Analysis of some finite games

Prisoner's Dilemma Recall that the prisoner's dilemma can be described by the following matrix:

		University II	
		Give him 2	Give me 1
University I	Give him 2	2, 2	0, 3
	Give me 1	3, 0	1, 1

The action profile (*Give me 1, Give me 1*) is a Nash equilibrium. Indeed,

$$u_I(\textit{Give me 1, Give me 1}) = 1 \geq u_I(\textit{Give him } 2, \textit{Give me } 1) = 0$$

and

$$u_{II}(\textit{Give me } 1, \textit{Give me } 1) = 1 \geq u_{II}(\textit{Give me } 1, \textit{Give him } 2) = 0.$$

On the other hand, the action profile (*Give him 2, Give him 2*) is not a Nash equilibrium, since University I prefers action "*Give me 1*" if University II chooses action "*Give him 2*":

$$2 = u_I(\textit{Give him } 2, \textit{Give him } 2) < u_I(\textit{Give me } 1, \textit{Give him } 2) = 3.$$

Battle of the Sexes Recall that the battle of the sexes can be described by the following matrix:

		She	
		Box	Ballet
He	Box	2, 1	0, 0
	Ballet	0, 0	1, 2

One can check that (*Box, Box*) is a Nash equilibrium and (*Ballet, Ballet*) is a Nash equilibrium as well. It can also be checked that these are the only two action profiles that constitute a Nash equilibrium.

Matching Pennies The reader can check that Matching Pennies has no Nash equilibrium.

Before we analyze the next example, we introduce a technical tool that allows us to reformulate the definition of Nash equilibrium more conveniently. More importantly, this alternative definition is the key to the standard proof of the existence of Nash equilibrium.

Definition 3 Let $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be a strategic game and let $i \in N$ be a player. Consider a list of actions $a_{-i} = (a, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) \in \times_{k\in N\setminus\{i\}} A_k$ of all the players other than i. The set of player i's best responses to a_{-i} is

$$\mathcal{B}_i(a_{-i}) = \{a_i \in A_i : \; u_i(a_i, a_{-i}) \geq u_i(b_i, a_{-i}) \quad \text{for all } b_i \in A_i\}.$$

The correspondence $\mathcal{B}_i : \times_{j\neq i} A_j \twoheadrightarrow A_i$ that assigns to each $(n-1)$-tuple of actions in A_{-i} the set of best responses to it is called the *best response correspondence* of player i.

The definition of a Nash equilibrium may be stated in terms of the players' best response correspondences, as stated in the following proposition.

Proposition 1 The action profile $a^* \in A$ is a Nash equilibrium if and only if every player's action is a best response to the other players' actions. That is, if

$$a_i^* \in \mathcal{B}_i(a_{-i}^*) \qquad \text{for all } \; i \in N.$$

Until now, all the examples involved games where the action sets contained two actions. The next example is a game where the players' action sets are infinite. We will use the player's best response correspondences to find all its Nash equilibria.

The War of Attrition Two animals, 1 and 2, are fighting over a prey. Each animal chooses a time at which it intends to give up. Once one animal has given up, the other obtains the prey; if both animals give up at the same time then they split the prey equally. For each $i = 1, 2$, animal i's willingness to fight for the prey is given by $v_i > 0$. The value v_i is the maximum amount of time that animal i is willing to spend to obtain the prey. Since fighting is costly, each animal prefers as short a fight as possible. If animal i obtains the prey after a fight of length t, his utility will be $v_i - t$. We can model the situation as the game $G = \langle\{1,2\}, (A_1, A_2), (u_1, u_2)\rangle$ where

- $A_1 = [0, \infty] = A_2$ (an element $t \in A_i$ represents a time at which player i plans to give up)
- $u_1(t_1, t_2) = \begin{cases} -t_1 & \text{if } t_1 < t_2 \\ \frac{1}{2}v_1 - t_2 & \text{if } t_1 = t_2 \\ v_1 - t_2 & \text{if } t_1 > t_2 \end{cases}$
- $u_2(t_1, t_2) = \begin{cases} -t_2 & \text{if } t_2 < t_1 \\ \frac{1}{2}v_2 - t_1 & \text{if } t_1 = t_2 \\ v_2 - t_1 & \text{if } t_2 > t_1 \end{cases}$

We are interested in the best response correspondences. First, we calculate player 1's best response correspondence, $\mathcal{B}_1(t_2)$. There are three cases to consider.

Case 1: $t_2 < v_1$**.** In this case, $v_1 - t_2 > \frac{1}{2}v_1 - t_2$ and $v_1 - t_2 > -t_1$. Consequently, given that player 2's action is t_2, player 1's utility function has a maximum value of $v_1 - t_2$, which is attained at any $t_1 > t_2$. Therefore, $\mathcal{B}_1(t_2) = (t_2, \infty)$.

Case 2: $t_2 = v_1$**.** In this case, $0 = v_1 - t_2 > \frac{1}{2}v_1 - t_2$. Therefore, player's 1 utility function $u_1(\cdot, t_2)$ has a maximum value of 0, which is attained at $t_1 = 0$ and at $t_1 > t_2$. Therefore, $\mathcal{B}_1(t_2) = \{0\} \cup (t_2, \infty)$.

Case 3: $t_2 > v_1$**.** In this case $\frac{1}{2}v_1 - t_2 < v_1 - t_2 < 0$. As a result, player 1's utility function $u_1(\cdot, t_2)$ has a maximum value of 0, which is attained at $t_1 = 0$. Therefore, $\mathcal{B}_1(t_2) = \{0\}$.

Summarizing, player 1's best response correspondence is:

$$\mathcal{B}_1(t_2) = \begin{cases} (t_2, \infty) & \text{if } t_2 < v_1 \\ \{0\} \cup (t_2, \infty) & \text{if } t_2 = v_1 \\ \{0\} & \text{if } t_2 > v_1 \end{cases}$$

which is depicted in Figure 1.

Figure 1 about here

Similarly, player 2's best response correspondence is:

$$\mathcal{B}_2(t_1) = \begin{cases} (t_1, \infty) & \text{if } t_1 < v_2 \\ \{0\} \cup (t_1, \infty) & \text{if } t_1 = v_2 \\ \{0\} & \text{if } t_1 > v_2 \end{cases}$$

Combining the two best response correspondences we get that (t_1^*, t_2^*) is a Nash equilibrium if and only if either $t_1^* = 0$ and $t_2^* \geq v_1$ or $t_2^* = 0$ and $t_1^* \geq v_2$. Figure 2 depicts the set of all the Nash equilibria as the intersection of the two best response correspondences.

Figure 2 about here

Two things are worth noting. First, it is not necessarily the case that the player who values the prey most wins the war. That is, there are Nash equilibria of the war of attrition where the player with the highest willingness to fight for the prey gives in first, and as a result the object goes to the other player. Second, in none of the Nash equilibria is there a physical fight. All Nash equilibria involve one player giving in immediately to the other. This second feature seems rather unrealistic, since fights in "war of attrition"-like situations are commonly observed. If one wants to obtain a fight of positive length in the war of attrition one needs to either drop the Nash equilibrium concept and adopt an alternative one, or model the war of attrition differently. We will adopt this second course of action later.

4 Existence

As the matching pennies example shows, not all games have a Nash equilibrium. The following theorem, which dates back to Nash (1950) and Glicksberg (1952), states sufficient conditions on a game for it to have a Nash equilibrium. An earlier version of this theorem for the smaller but prominent class of zero-sum games can be found in von Neumann

(1928) (translated in von Neumann (1959)). The standard proofs use Kakutani's fixed point theorem. We present here an alternative proof, due to Geanakoplos (2003), which uses Brouwer's fixed point theorem instead.

Theorem 1 The game $\langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ has a Nash equilibrium if for all $i \in N$

- the set A_i of actions of player i is a nonempty compact convex subset of an Euclidean space,
- the utility function u_i is continuous,
- the utility function u_i is concave in A_i.

Proof : (Geanakoplos) Define the correspondence $\varphi_i : A \twoheadrightarrow A_i$ by

$$\varphi_i(\bar{a}) = \arg\max_{a_i \in A_i}\{U_i(a_i, \bar{a}_{-i}) - \|a_i - \bar{a}_i\|^2\},$$

where, $\|\cdot\|$ denotes a norm in the relevant Euclidean space. Note first that φ_i is a nonempty valued correspondence because the maximand is a continuous function and A_i is compact. Second, note that the function $\|a_i - \bar{a}_i\|$ is convex:

$$\begin{aligned}
\|(\lambda a_i + (1-\lambda)b_i) - \bar{a}_i\| &= \|(\lambda a_i - \lambda\bar{a}_i) + ((1-\lambda)b_i - (1-\lambda)\bar{a}_i)\| \\
&\leq \|(\lambda a_i - \lambda\bar{a}_i)\| + \|((1-\lambda)b_i - (1-\lambda)\bar{a}_i)\| \\
&\leq |\lambda|\|a_i - \bar{a}_i\| + |1-\lambda|\|b_i - \bar{a}_i\|.
\end{aligned}$$

Since the quadratic function is strictly convex, then the maximand is a strictly concave function. Therefore, the correspondence φ_i is in fact a function. Furthermore, since the maximand is continuous in the parameter $\bar{a}$, φ_i is also continuous. To see this, let $\bar{a}_n \to \bar{a}$ be a convergent sequence of action profiles and let $a_{in} = \varphi_i(\bar{a}_n)$. This means that $U(a_{in}, (\bar{a}_n)_{-i}) \geq U(b_i, (\bar{a}_n)_{-i})$ for all $b_i \in A_i$. Since A_i is a compact set, a_{in} has a convergent subsequence. Denoting by a_i the limit of this subsequence and applying limits to the above inequality, we obtain that

$$U(a_i, \bar{a}_{-i}) \geq U(b_i, \bar{a}_{-a}) \qquad \text{for all } b_i \in A_i,$$

namely $a_i = \varphi_i(\bar{a})$. Since this is true for every convergent subsequence of a_{in}, we have that $\varphi_i(\bar{a}_n) = a_{in} \to a_i = \varphi_i(\bar{a})$, which means that φ is continuous.

Now define $\varphi : A \to A$ by $\varphi = (\varphi_1, \ldots, \varphi_N)$. Clearly, φ is a continuous function mapping a compact set to itself. Therefore, by Brouwer's fixed point theorem, it has a fixed point: $\varphi(\bar{a}) = \bar{a}$. We now show that $\bar{a}$ is a Nash equilibrium of the game. Assume not. Then, there is some $i \in N$ with $a_i \in A_i$ such that $U_i(a_i, \bar{a}_{-i}) - U_i(\bar{a}) = E > 0$. Then, by concavity of U_i, for all $0 < \epsilon < 1$,

$$\begin{aligned} U_i(\epsilon a_i + (1-\epsilon)\bar{a}_i, \bar{a}_{-i}) - U_i(\bar{a}) &\geq \epsilon U_i(a_i, \bar{a}_{-i}) + (1-\epsilon)U_i(\bar{a}) - U_i(\bar{a}) \\ &\geq \epsilon E > 0, \end{aligned}$$

while $\|\epsilon a_i + (1-\epsilon)\bar{a}_i - \bar{a}_i\|^2 = \epsilon^2 \|a_i - \bar{a}_i\|^2 < \epsilon E$, for small enough ϵ. Therefore, for such small ϵ, the action $\epsilon a_i + (1-\epsilon)\bar{a}_i$ satisfies

$$U_i(\epsilon a_i + (1-\epsilon)\bar{a}_i, \bar{a}_{-i}) - \|\epsilon a_i + (1-\epsilon)\bar{a}_i - \bar{a}_i\|^2 > U_i(\bar{a})$$

which contradicts the fact that $\varphi_i(\bar{a}) = \bar{a}_i$. □

5 Mixed Strategies

So far, we have formally defined a game, and have introduced the solution concept of Nash equilibrium which is arguably the central solution concept of game theory. However, there seem to be two problems with this concept. One is that although Nash equilibria exist in a wide class of games, there are many simple games that do not have a Nash equilibrium. The most troubling example is Matching Pennies. If game theory cannot provide a prediction for this simple game then one must wonder if there is any value to the theory. The second problem is that the concept of Nash equilibrium predicts a very unrealistic outcome in the war of attrition. One would expect that game theory would not only provide nonempty predictions, but also ones that look reasonable and help explain what we see around us.

One way to approach these problems is not to abandon the theory or the concept of Nash equilibrium altogether, but to modify the way we model the problematic situations. The idea behind mixed strategies is to first modify the game by extending the set of actions available to the players, and then to apply the concept of Nash equilibrium to this extended game. In this way one may obtain additional Nash equilibria, some of which may provide reasonable predictions to the game.

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a finite game. For any A_i, a probability distribution on A_i is a function

$$x_i : A_i \to \mathbb{R}_+$$

such that

$$\sum_{a_i \in A_i} x_i(a_i) = 1.$$

The set of all probability distributions on A_i is denoted by $\Delta(A_i)$. A *mixed strategy on A_i* is a random choice over elements of A_i, namely an element of $\Delta(A_i)$. If x_i is a mixed strategy on A_i, $x_i(a_i)$ denotes the probability that action $a_i \in A_i$ is selected when x_i is adopted. Since elements of $\Delta(A_i)$ can have an alternative interpretation, such as beliefs about the choice of player i, we denote the set of mixed strategies by X_i to distinguish it from the more abstract set of probability distributions on A_i. Also, we denote the set of mixed strategy profiles as $X = \times_{i \in N} X_i$. Denoting for each player $i \in N$, $X_{-i} = \times_{k \in N \setminus \{i\}} X_k$, a typical mixed strategy profile can be written as $(x_k)_{k \in N} = (x_i, x_{-i}) \in X_i \times X_{-i}$. The *mixed extension* of the strategic game G is the strategic game $\langle N, (X_i)_{i \in N}, (U_i)_{i \in N} \rangle$ where the set of actions of player i is the set of mixed strategies, X_i, and the payoff function

$U_i : \times_{i\in N} X_i \to \mathbb{R}$ of player i is defined by

$$U_i((x_k)_{k\in N}) = \sum_{a=(a_k)_{k\in N}\in A} u_i(a)\Pi_{k\in N} x_k(a_k)$$

Remark. Since each mixed strategy of player i, x_i, can be identified with a vector $x_i = (x_i(a_i))_{a_i\in A_i} \in \mathbb{R}^{|A_i|}$, the function U_i is multinomial in the coordinates of its variables, and, as a result, it is continuous as a function of the players' mixed strategies.

Definition 4 An *equilibrium in mixed strategies* of the game $\langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ is a Nash equilibrium of the mixed extension of the game. In other words, it is a list of mixed strategies $(x_k^*)_{k\in N} \in X$ such that for all players $i \in N$ and for all his mixed strategies x_i,

$$U_i((x_k^*)_{k\in N}) \geq U_i(\left(x_i, x_{-i}^*\right)).$$

Alternatively, $(x_k^*)_{k\in N} \in X$ is a mixed strategy equilibrium if

$$x_i^* \in \mathcal{B}_i(x_{-i}^*) \qquad \text{for all} \quad i \in N.$$

Note that for every finite game $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$, its mixed extension is a strategic game that satisfies the conditions of Theorem 1. As a result, every finite game has a mixed strategy equilibrium.

Example 1 Consider again Matching Pennies. Its mixed extension is the game $\langle N, (X_i)_{i\in N}, (U_i)_{i\in N}\rangle$ where the set of players is $N = \{1, 2\}$, the sets of mixed strategies are $X_1 = \{(p_H, p_T) \geq (0,0) : p_H + p_T = 1\}$, and $X_2 = \{(q_H, q_T) \geq (0,0) : q_H + q_T = 1\}$, and the utility functions are given by $U_1((p_H, p_T), (q_H, q_T)) = p_H q_H + p_T q_T - p_H q_T - p_T q_H$ and $U_2((p_H, p_T), (q_H, q_T)) = p_H q_T + p_T q_H - p_H q_H - p_T q_T$. It can be checked that the only Nash equilibrium of this mixed extension is $((1/2, 1/2), (1/2, 1/2))$. Indeed, since $U_1((p_H, p_T), (1/2, 1/2))$ is identically 0, it attains its maximum at, among other strategies, $(1/2, 1/2)$. The same is true for $U_2((1/2, 1/2), (q_H, q_T))$. To see that there is no other equilibrium, note that for (q_H, q_T) with $q_H > q_T$, player 1's best response is $(1, 0)$. But player 2's best response to $(1,0)$, is $(0,1)$. Since $0 \leq 1$, (q_H, q_T) with $q_H > q_T$ cannot be part of an equilibrium. Similarly, for any (q_H, q_T) with $q_H < q_T$, player 1's best response is $(0,1)$. But player 2's best response to $(0,1)$ is $(1,0)$. Since $1 \geq 0$, (q_H, q_T) with $q_H < q_T$ cannot be part of an equilibrium.

We next present a characterization of the mixed strategy equilibria of a game that will sometimes allow us to compute them in an easy way. Further, this characterization serves as the basis of an interesting interpretation of the mixed strategy equilibrium concept that we will discuss later. For this purpose, we identify the action $a_i \in A_i$ of player i with the mixed strategy of player i that assigns probability 1 to action a_i, and 0 to all other actions. Therefore, given a player i, one of his actions $a_i \in A_i$, and a profile $x = (x_k)_{k \in N}$ of the players' mixed strategies, (a_i, x_{-i}) denotes the mixed strategy profile obtained from x by replacing i's mixed strategy x_i by the mixed strategy of player i that assigns probability 1 to action a_i. With this notation we can state the following identity:

$$U_i((x_k)_{k \in N}) = \sum_{a_i \in A_i} x_i(a_i) U_i((a_i, x_{-i})). \tag{1}$$

Indeed,

$$\begin{aligned}
U_i((x_k)_{k \in N}) &= \sum_{a=(a_k)_{k \in N} \in A} u_i(a) \Pi_{k \in N} x_k(a_k) \\
&= \sum_{a_i \in A_i} \sum_{a_{-i} \in A_{-i}} u_i(a) \Pi_{k \in N} x_k(a_k) \\
&= \sum_{a_i \in A_i} x_i(a_i) \sum_{a_{-i} \in A_{-i}} u_i(a) \Pi_{k \in N \setminus \{i\}} x_k(a_k) \\
&= \sum_{a_i \in A_i} x_i(a_i) U_i((a_i, x_{-i})).
\end{aligned}$$

Identity (1) is useful to prove the following characterization of the mixed strategy Nash equilibria.

Lemma 1 The strategy profile $x^* = (x_k^*)_{k\in N}$ is an equilibrium of the mixed extension of $\langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ if and only if for all players $i \in N$ and for all $a_i \in A_i$,

$$\text{If } x_i^*(a_i) > 0 \text{ then } U_i\big(\big(a_i, x_{-i}^*\big)\big) = U_i(x^*) \tag{2}$$

$$\text{If } x_i^*(a_i) = 0 \text{ then } U_i\big(\big(a_i, x_{-i}^*\big)\big) \le U_i(x^*). \tag{3}$$

Proof : Assume that $x^* = (x_k^*)_{k\in N}$ satisfies conditions (2) and (3). Let $i \in N$, and let x_i be a mixed strategy of player i. Then, by (1)

$$U_i(x_i, x_{-i}^*) = \sum_{a_i\in A_i} x_i(a_i) U_i\big(\big(a_i, x_{-i}^*\big)\big) \le \sum_{a_i\in A_i} x_i(a_i) U_i(x^*) = U_i(x^*)$$

and therefore x^* is an equilibrium.

Assume now that $x^* = (x_k^*)_{k\in N}$ is an equilibrium. Let $i \in N$. Then

$$U_i(x^*) \ge U_i\big(\big(a_i, x_{-i}^*\big)\big) \qquad \forall a_i \in A_i \tag{4}$$

and, in particular, condition (3) holds for all $a_i \in A_i$ such that $x_i(a_i) = 0$. Also, using (1) we can write

$$\sum_{a_i\in A_i} x_i^*(a_i) U_i(x^*) = \sum_{a_i\in A_i} x_i^*(a_i) U_i\left(a_i, x_{-i}^*\right). \tag{5}$$

If there is $a_i \in A_i$ such that $x_i(a_i) > 0$ and $U_i(x^*) > U_i\left(a_i, x_{-i}^*\right)$ then, using (4),

$$\sum_{a_i\in A_i} x_i^*(a_i) U_i(x^*) > \sum_{a_i\in A_i} x_i^*(a_i) U_i\big(\big(a_i, x_{-i}^*\big)\big)$$

in contradiction to (5). $\square$

Corollary 1 The strategy profile $x^* = (x_k^*)_{k\in N}$ is an equilibrium of the mixed extension of $\langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ if and only if for all players $i \in N$ and for all $a_i \in A_i$,

$$x_i^*(a_i) > 0 \text{ implies } a_i \in \mathcal{B}_i(x_{-i}^*).$$

According to the standard interpretation, a player's mixed strategy in a game G is an action, but in a different game, namely in the mixed extension of G. According to this interpretation, a mixed strategy is a deliberate choice of a player to use a random device. A mixed strategy equilibrium then is a profile of *independent* random devices, each of which is a best response to the others. Corollary 1 provides an alternative interpretation of a mixed strategy equilibrium. According to this interpretation, a player's mixed strategy represents the uncertainty in the minds of the other players concerning the player's action. In other words, a player's mixed strategy is interpreted not as a deliberate choice of the player but the belief, shared by all the other players, about the player's choice. That is, if $(x_k)_{k\in N}$ is a profile of mixed strategies, then x_i is the conjecture, shared by all the players other than i, about i's ultimate choice of action. Consequently, x_{-i} are the conjectures entertained by player i about his opponents' actions. According to this interpretation, Corollary 1 says that a mixed strategy equilibrium $(x_k^*)_{k\in N}$ is a profile of beliefs about each player's actions (entertained by the other players) according to which each player chooses an action that is a best response to his own beliefs.

5.1 The War of Attrition (cont.)

We have seen in Section 3.1 that all the Nash equilibria of the war of attrition predict no real fight for the prey. We will now see that there is a mixed strategy equilibrium of the war of attrition that predicts a positive-length fight with probability one.

The players' action sets in the war of attrition are intervals of real numbers. A mixed strategy for player i in that game can be represented by a cumulative distribution function $F_i : [0,\infty] \to [0,1]$. For each $t \in (0,\infty]$, $F_i(t)$ is the probability that player i gives up at or before t. We will look for a Nash equilibrium (F_1, F_2) that consists of two strictly increasing, differentiable cumulative distribution functions. The density of F_i is denoted by f_i. We will try to find an equilibrium at which each player is indifferent between all pure actions.

Consider player i. Given that his opponent is using mixed strategy F_j, $j \neq i$, if he chooses to give in at time t, then he will face a lottery according to which,

- with probability $1 - F_j(t)$, player i does not obtain the prey and gets a payoff of $-t$,
- with probability $F_j(t)$, player i obtains the prey at time t_j, where t_j is a random variable whose cumulative distribution function is $F_j(t_j)/F_j(t)$ (the distribution player j's surrender time, conditional on his having given in before t).

Therefore, the corresponding expected utility of choosing time t is

$$\begin{aligned} U_i(t, F_j) &= (1 - F_j(t))(-t) + F_j(t) \int_0^t (v_i - t_j) d\frac{F_j(t_j)}{F_j(t)} \\ &= (1 - F_j(t))(-t) + \int_0^t (v_i - t_j) dF_j(t_j). \end{aligned}$$

Since in the equilibrium we are looking for, player i is indifferent among all his actions, the above expression is independent of t. Namely, $U_i(t, F_j) \equiv c$. As a result, the derivative of the above utility with respect to t equals 0. Formally,

$$\begin{aligned} \frac{\partial U_i(t, F_j)}{\partial t} &= t f_j(t) - (1 - F_j(t)) + (v_i - t) f_j(t) \\ &= (1 - F_j(t)) + v_i f_j(t) = 0. \end{aligned}$$

This is a differential equation whose general solution is

$$F_j(t) = 1 - K e^{-\frac{t}{v_i}}.$$

If we want it to satisfy $F_j(0) = 0$, we obtain that $K = 1$. As a result, the distribution function is given by

$$F_j(t) = 1 - e^{-\frac{t}{v_i}}.$$

Consequently, the equilibrium we are looking for is

$$(F_1(t), F_2(t)) = (1 - e^{-\frac{t}{v_2}}, 1 - e^{-\frac{t}{v_1}}).$$

According to this equilibrium, for any t, the probability that there is a fight that lasts at least t is $(1-F_1(t))(1-F_2(t)) > 0$. Consequently, there is a fight with probability one. The introduction of mixed strategies allowed the concept of Nash equilibrium to be consistent with fights that last a positive length of time. However, the mixed strategy equilibrium has the following unfortunate property. If $v_1 < v_2$, then for all $t > 0$, $F_1(t) < F_2(t)$. In other words, it is more likely that the player with the highest willingness to fight for the prey gives up earlier than any given t, than that the player with the lowest willingness to fight gives in earlier than the same t. Therefore, in equilibrium it is more likely that the player with the lower willingness to fight wins the war than the other way around. In particular, the probability that player 1 gets the object is given by

$$\int_0^\infty F_2(t)\ dF_1(t)$$

which can be checked to be equal to $\frac{v_2}{v_1+v_2} > 1/2$. In order to obtain the more intuitive result that the higher the willingness to fight for the prey, the higher is the probability to obtain it, we will need to model the war of attrition in yet a different way. We'll return to this when we introduce asymmetric information to the games.

6 Equilibrium in Beliefs

The mixed extension of the game $\langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ is constructed in two steps. First, we enlarge the set of actions available to each player by allowing him to choose any mixed strategy on his original action set. Second, since the action choices are now probability distributions over actions, we extend the players' original preferences to preferences over profiles of mixed strategies. We do so by evaluating each mixed strategy profile according to the expected value of the original utilities with respect to the probability distribution over action profiles induced by the mixed strategy.

The first step seems uncontroversial since it is certainly possible for players to use random devices. But the second step is somewhat problematic because, by evaluating mixed strategies according to the expected utility of the resulting lotteries, one is implicitly imposing on the players a certain kind of risk preferences. One may wonder what the implications would be if instead of extending the preferences by assuming that players are expected utility maximizers, we assume that players have more general preferences over profiles of mixed strategies. In particular, we would like to know if there is a suitable generalization of Corollary 1.

Let $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be a finite game. We define the mixed extension of G as the strategic game $\langle N, (X_i)_{i\in N}, (U_i)_{i\in N}\rangle$ where, as in Section 5, X_i is the set of probability distributions over the actions in A_i, for $i \in N$, but unlike there, the utility function $U_i : X \to \mathbb{R}^N$ is not necessarily a multilinear function of the probabilities, but a general continuous function of the mixed strategies. The only requirement on U_i is that for all profiles of degenerate mixed strategies $(a_k)_{k\in N}$, we have $U_i\left((a_k)_{k\in N}\right) = u_i\left((a_k)_{k\in N}\right)$. As before, a mixed strategy Nash equilibrium of $\langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ is a Nash equilibrium of its mixed extension $\langle N, (X_i)_{i\in N}, (U_i)_{i\in N}\rangle$. In other words, it is a list of mixed strategies $(x_k^*)_{k\in N}$ such that for all players $i \in N$ and for all of his mixed strategies x_i,

$$U_i((x_k^*)_{k\in N}) \geq U_i(\left(x_i, x_{-i}^*\right)).$$

Alternatively, $(x_k^*)_{k\in N}$ is a mixed strategy equilibrium if

$$x_i^* \in \mathcal{B}_i(x_{-i}^*) \qquad \text{for all } \ i \in N.$$

Observation 1 It is important to note that two different actions of a player may be best responses to a given mixed strategy profile of the other players, and yet no probability mixture of the two actions will be a best response to the given mixed strategy profile. This will typically be the case when the function U_i is strictly convex in X_i, since strictly convex functions attain their maximum at boundary points.

Theorem 1 shows that Nash equilibria exist when the extended utility function U_i is concave in X_i. However, Observation 1 indicates that a Nash equilibrium may fail to exist when U_i is strictly convex in X_i. Indeed, take a game $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ with no pure strategy Nash equilibrium, like Matching Pennies, and consider its mixed extension $\Gamma = \langle N, (X_i)_{i\in N}, (U_i)_{i\in N}\rangle$ where for all players, their extended utility function is strictly convex. Then, for any player $i \in N$ and for any profile of mixed strategies x_{-i} of the other players, the set of i's best responses $\mathcal{B}_i(x_{-i})$ consists of only degenerate mixed strategies. Since G has no pure strategy Nash equilibrium, we conclude that Γ does not have a Nash equilibrium.

Observation 2 It is also important to note that, unlike in the standard expected utility case, a player's mixed strategy x_i^* may very well be a best response to some profile x_{-i}^* of the other players' mixed strategies and at the same time may assign positive probability to an action that (when regarded as a degenerate mixed strategy) is not a best response to x_{-i}^*. Formally, it may very well be the case that

$$U_i((x_k^*)_{k\in N}) \geq U_i(\big(x_i, x_{-i}^*\big)) \qquad \text{for all } x_i \in X_i$$

and yet

$$U_i(\big(a_i, x_{-i}^*\big)) < U_i((x_k^*)_{k\in N}) \qquad \text{for some } a_i \text{ such } x_i^*(a_i) > 0.$$

This will typically occur when the function U_i is strictly concave in X_i.

The definition of mixed strategy equilibrium requires from each strategy in the equilibrium profile that it be a best response to the other strategies. Corollary 1 stated that when preferences have the expected utility form, each mixed strategy in a mixed strategy equilibrium is also a probability mixture over best responses to the other strategies in the profile. This result allowed us to interpret a mixed strategy Nash equilibrium as a profile of beliefs, rather than as a profile of probability mixtures. As explained in Observation 2, however, when preferences over mixed strategies are not expected utility preferences, a mixture over best responses is not necessarily a best response. Therefore, Corollary 1 does not extend to the mixed extension where preferences are not of the expected utility form.

In this setup, however, one can still interpret a player's mixed strategy as a belief entertained by the other players about the actions chosen by that player. And a profile of such beliefs will be in equilibrium if the probability distribution over the player's actions that represents i's beliefs is obtained as a mixture of best responses of this player to his beliefs about the other players' actions. With this idea in mind, Crawford (1990) defined the notion of an equilibrium in beliefs. Before we formally present his definition we need to introduce some notation.

Since when the extended utility functions U_i are concave in i's own strategy a best response to a given profile of the other players' strategies may be a non-degenerate mixed strategy, a mixture of best responses will typically be a mixture over non-degenerate mixed strategies. This mixture induces a probability distribution over actions in a natural way by reducing the compound mixture to a simple mixture. This induced probability distribution can be interpreted as a belief over the actions ultimately chosen. For example, in Matching Pennies, if player 1 believes that there is a probability of 1/2 that player 2 will choose the mixed strategy $(1/3, 2/3)$ and a probability of 1/2 that player 2 will choose the mixed strategy $(2/3, 1/3)$, then player 1 believes that player 2 will choose each one of his two actions with equal probability. More generally, if player i assigns probability p_k to the event that player j will choose mixed strategy $x^k \in X_j$, for $k = 1, \dots K$, then player i's beliefs about player j's actions are given by $\sum_{k=1}^{K} p_k x^k \in X_j$. That is, for each action $a_j \in A_j$ of player j, player i believes that player j will choose a_j with probability $\sum_{k=1}^{K} p_k x^k(a_j)$. For each set $T \subset X_i$ of mixed strategies, let $D[T] \subset X_i$ denote the set of probability distributions over i's actions that are induced by mixtures over elements of T. With this notation in hand, we can define the concept of equilibrium in beliefs.

Definition 5 Let $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be a game. For each $i \in N$, let $\mathcal{B}_i : X \to X_i$ be the best response correspondence in the mixed extension of G. The profile of beliefs $(x_k^*)_{k\in N} \in \times_{k\in N}\Delta(A_k)$ is an *equilibrium in beliefs* if

$$x_i^* \in D\left[\mathcal{B}_i(x_{-i}^*)\right] \qquad \text{for all} \qquad i \in N.$$

An equilibrium in beliefs is a profile of beliefs $(x_k^*)_{k\in N}$. For each $i \in N$, x_i^* is the common belief of the players other than i about player i's choice of actions. In order for this profile of beliefs to be in equilibrium, we require that for each player $i \in N$ all the other players believe that i chooses a mixed strategy that is a best response to his beliefs, which are given by $(x_k^*)_{k\in N\setminus\{i\}}$, about the other players' choices of actions. In other words, x_i^* must be a convex combination of best responses of i to $(x_k^*)_{k\in N\setminus\{i\}}$.

Example 2 Consider again the mixed extension of Matching Pennies $\langle N, (X_i)_{i\in N}, (U_i)_{i\in N}\rangle$ where the set of players is $N = \{1,2\}$, the sets of mixed strategies are $X_1 = \{(p_H, p_T) \geq (0,0) : p_H + p_T = 1\}$ and $X_2 = \{(q_H, q_T) \geq (0,0) : q_H + q_T = 1\}$, and the utility functions are now given by $U_1((p_H,p_T),(q_H,q_T)) = (p_H q_H)^2 + (p_T q_T)^2 - p_H q_T - p_T q_H$ and $U_2((p_H,p_T),(q_H,q_T)) = (p_H q_T)^2 + (p_T q_H)^2 - p_H q_H - p_T q_T$. Since the utility functions are strictly convex in the players's own mixed strategies, the best response to any strategy of the opponent is a pure strategy. In particular, one can verify that

$$\mathcal{B}_1(q_H, q_T) = \begin{cases} (1,0) & \text{if } q_H > q_T \\ \{(1,0),(0,1)\} & \text{if } q_H = q_T \\ (0,1) & \text{if } q_H < q_T \end{cases}$$

and

$$\mathcal{B}_2(p_H, p_T) = \begin{cases} (0,1) & \text{if } p_H > p_T \\ \{(1,0),(0,1)\} & \text{if } p_H = p_T \\ (1,0) & \text{if } p_H < p_T. \end{cases}$$

It can also be verified that $((p_H^*, p_T^*),(q_H^*, q_T^*)) = ((1/2,1/2),(1/2,1/2))$ is an equilibrium in beliefs. Indeed, for both $i = 1,2$, $(1/2,1/2) \in X_i$ is a convex combination of $(1,0)$ and $(0,1)$, which are both in $\mathcal{B}_j(1/2,1/2)$, $j \neq i$. In this equilibrium,

1. Player 1 believes that player 2 will choose $(1,0)$ with probability $1/2$, and $(0,1)$ with probability $1/2$.

2. Therefore player 1 believes that player 2 will ultimately choose H and T, each with probability $1/2$.

3. Given these beliefs, player 1's only best replies are $(1,0)$ and $(0,1)$, and
4. Player 2 believes that player 1 will choose each one with probability 1/2. As a result,
5. Player 2 believes that player 1 will ultimately choose H and T each with probability 1/2.
6. Given these beliefs, player 2's only best replies are $(1,0)$ and $(0,1)$, and
1. Player 1 believes that player 2 will choose $(1,0)$ with probability 1/2, and $(0,1)$ with probability 1/2.

The following result is a direct implication of the definition of an equilibrium in beliefs.

Proposition 2 [Crawford (1990)] Let $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be a strategic game, and $\Gamma = \langle N, (X_i)_{i\in N}, (U_i)_{i\in N}\rangle$ be the mixed extension of G, where U_i is continuous but not necessarily multilinear.

1. Every mixed strategy Nash equilibrium of G is an equilibrium in beliefs.
2. If for all $i \in N$, U_i is quasiconcave in X_i, then every equilibrium in beliefs is a mixed strategy Nash equilibrium of G.

Proof :

1. Since $\mathcal{B}_i(x^*_{-i}) \subset D\left[\mathcal{B}_i(x^*_{-i})\right]$ for all $i \in N$, every Nash equilibrium is an equilibrium in beliefs.
2. When the utility function U_i is quasiconcave in i's mixed strategy, the set of best responses $\mathcal{B}_i(x^*_{-i})$ is a convex set. Therefore, $D\left[\mathcal{B}_i(x^*_{-i})\right] = \mathcal{B}_i(x^*_{-i})$, and any equilibrium in beliefs is a Nash equilibrium.

□

Crawford (1990) shows that although some games have no Nash equilibrium, every game has an equilibrium in beliefs.

7 Correlated Equilibrium

In the mixed extension of a game, players do not choose their actions directly, but rather choose probability distributions over their action sets according to which the actions are ultimately selected. The important feature about these probability distributions is that they represent independent random variables. The realization of one player's random variable does not give any information about the realization of the other players' random variables. There is nothing in the bare notion of equilibrium, however, that requires players' behavior to be independent. The basic feature of an equilibrium is that each player is best responding to the behavior of others, and that each player is free to choose any action in his action set. But one thing is that players can, if they so wish, change their behavior without the consent of others, and another different thing is to expect players' choices to be independent. Therefore, one could ask what would happen if the random devices players use to ultimately choose their actions were correlated. In that case, knowledge of the realization of one's random device would provide some partial information about the realization of the other players' random devices, and therefore of their choices. In equilibrium, a player should take this information into account. To illustrate this point, consider the game of Chicken.

		Driver 2	
		Slow Down	Speed up
Driver 1	Slow Down	6, 6	2, 7
	Speed up	7, 2	0, 0

This game has two pure-action Nash equilibria, and one equilibrium in mixed strategies. According to the mixed strategy Nash equilibrium, each player chooses Slow Down with probability 2/3 and Speed Up with probability 1/3. This mixed strategy equilibrium can be implemented by the following random device. Consider two random variables S_1 and S_2, whose joint distribution is given by the following table:

		S_2	
		1	2
S_1	1	4/9	2/9
	2	2/9	1/9

Table 1: A random device.

Driver 1 chooses his action as a function of the realization of S_1 and Driver 2 chooses his action as a function of the realization of S_2. (Neither player is informed of the realization of the other player's random variable.) In particular, Driver 1 chooses Slow Down if $S_1 = 1$ and Speed Up otherwise. Similarly, Driver 2 chooses Slow Down if $S_2 = 1$, and Speed Up otherwise. Note that according to this pattern of behavior, each player chooses to slow down with probability 2/3. But more importantly, since S_1 and S_2 are independent random variables, knowledge of the realization of one random variable does not give any information about the realization of the other one. Therefore, after Driver 1 learns the realization of S_1, he still believes that Driver 2 will choose Slow Down with probability 2/3 and consequently any choice is optimal, in particular the one described above. Similarly, after Driver 2 learns the realization of S_2, he still believes that Driver 1 will choose to slow down with probability 2/3, and his planned behavior continues to be optimal.

But what would happen if the joint distribution of S_1 and S_2, was not as presented in Table 1, but rather as follows?

		S_2	
		1	2
S_1	1	1/3	1/3
	2	1/3	0

To answer this question, assume that both players still choose their actions according to the previous pattern of behavior: Driver 1 chooses Slow Down if $S_1 = 1$, and Speed Up otherwise. The same holds for Driver 2. As a result, it is still true that each player chooses Slow Down with probability 2/3 and Speed Up with probability 1/3. However, since this time the conditioning random variables S_1 and S_2 are not independent, knowledge of the realization of S_1 affects the beliefs of Driver 1 about the probability with which Driver 2 chooses his actions. In particular, if $S_1 = 1$, Driver 1 updates his beliefs and assigns probability 1/2 to Driver 2 choosing either action, and consequently, Driver 1's only optimal action is Slow Down, which is precisely the choice dictated by the above pattern of behavior. Similarly, if $S_1 = 2$, Driver 1 should update his beliefs and assign probability one that Driver 2 will choose Slow Down. Consequently, Driver 1's best reply is to follow the above pattern of behavior and choose Speed Up. One can see that, given that the players know that the random variables S_1 and S_2 are correlated and they use this information accordingly, there is no incentive for either of them to deviate from the proposed pattern of behavior. Therefore, we can say that this pattern of behavior is an equilibrium. This notion of a correlated equilibrium was introduced in Aumann (1974). Before we give a formal definition we introduce the concept of a correlated strategy profile, which will play a central role not only in this section, but in the next one as well.

Definition 6 Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a game. A *correlated strategy profile in G* consists of

- A finite probability space (Ω, π)
- For each player $i \in N$, a partition $\mathcal{P}_i$ of Ω into events of positive probability

- For each player $i \in N$, a function $\sigma_i : \Omega \to A_i$ which is measurable with respect to $\mathcal{P}_i$.

A correlated strategy profile is a description of what players do and know while playing the game G. The collection $\left\langle(\Omega, \pi), (\mathcal{P}_i)_{i\in N}\right\rangle$ represents the random devices used by the players to ultimately choose their actions. The underlying probability space that governs the players' random devices is (Ω, π). Ω is the set of states, and for each state ω, $\pi(\omega)$ is the probability that ω occurs. For each $i \in N$, the partition $\mathcal{P}_i$ represents player i's information. Each element of the partition represents a different realization of the random device used by i to choose his action. Two states that belong to the same element of the partition $\mathcal{P}_i$ cannot be distinguished by i, while two states that belong to different partition cells can be distinguished by him. For each player i, $\sigma_i : \Omega \to A_i$ is the random variable that describes players i's choice of action, $\sigma_i(\omega)$ being the action chosen by him at state ω. The measurability of σ_i with respect to $\mathcal{P}_i$ formalizes the requirement that the actions chosen by player i depend only on his information about the state of the world. Therefore, for any two states that belong to the same element of his partition, the actions chosen by i at those states must be the same. That is, for any $\omega, \omega' \in P \in \mathcal{P}_i$ we have $\sigma_i(\omega) = \sigma_i(\omega')$.

For example, the correlated strategy profile described earlier for the game of chicken can be formalized as $\left\langle(\Omega, \pi), (\mathcal{P}_i)_{i\in N}, (\sigma_i)_{i\in N}\right\rangle$, where $N = \{I, II\}$, and

- $\Omega = \{(1, 1), (1, 2), (2, 1))\}$
- $\pi(\omega) = 1/3$ for all $\omega \in \Omega$
- $\mathcal{P}_I = \{\{(1, 1), (1, 2)\}, \{(2, 1)\}\}$ and $\mathcal{P}_{II} = \{\{(1, 1), (2, 1)\}\}, \{(1, 2)\}$
- $\sigma_I(\omega) = \begin{cases} \text{Slow Down} & \text{if } \omega \in \{(1, 1), (1, 2)\} \\ \text{Speed up} & \text{if } \omega \in \{(2, 1)\} \end{cases}$
- $\sigma_{II}(\omega) = \begin{cases} \text{Slow Down} & \text{if } \omega \in \{(1, 1), (2, 1)\} \\ \text{Speed up} & \text{if } \omega \in \{(1, 2)\} \end{cases}$

According to this correlated strategy profile, there are three equally likely states, and the players can distinguish only one component of the state, namely the realization of their random variable. The players' actions are described by the functions σ_I and σ_{II} which depend only on the respective player's information.

In what follows we denote by $\sigma : \Omega \to A$ the function that associates with each $\omega \in \Omega$ the action profile induced by the strategies σ_k, for $k \in N$. That is, $\sigma = (\sigma_k)_{k \in N}$. Also, for any $i \in N$, $\sigma_{-i} = (\sigma_k)_{k \in N \setminus \{i\}}$ so that $\sigma = (\sigma_{-i}, \sigma_i)$. We are interested in correlated strategy profiles in which no player benefits by altering his behavior. These special profiles are introduced in the following definition.

Definition 7 Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game. A *correlated equilibrium of* G is a correlated strategy $\left\langle (\Omega, \pi), (\mathcal{P}_i)_{i \in N}, (\sigma_i)_{i \in N} \right\rangle$ such that for every $i \in N$ and every function $\tau_i : \Omega \to A_i$ that is measurable with respect to $\mathcal{P}_i$,

$$\sum_{\omega \in \Omega} \pi(\omega) u_i(\sigma_{-i}(\omega), \sigma_i(\omega)) \geq \sum_{\omega \in \Omega} \pi(\omega) u_i(\sigma_{-i}(\omega), \tau_i(\omega)). \tag{6}$$

The value $v_i = \sum_{\omega \in \Omega} \pi(\omega) u_i(\sigma_{-i}(\omega), \sigma_i(\omega))$ is player i's correlated equilibrium payoff.

In a correlated strategy profile each player plans to condition his choice of action on the realization of a random variable, and the players' random variables may be correlated. A correlated strategy profile is a correlated equilibrium if no player can find an alternative way to condition his choice on the same random device, so that his expected utility is increased. Note that the player presumably chooses his strategy (his way to condition his actions on the outcomes of the random device) before he learns the realization of the device. Nonetheless, he evaluates the outcomes generated by the players' strategies by taking into account the precise correlation of the random devices on which outcomes players are conditioning their behavior.

Although strictly speaking mixed strategy Nash equilibria are not correlated equilibria, they do induce a correlated equilibrium distribution over action profiles. In order to state this claim, we need the following definition.

Definition 8 Let $\langle(\Omega, \pi), (\mathcal{P}_i, \sigma_i)_{i\in N}\rangle$ be a correlated strategy profile for G. Its induced probability distribution over action profiles is given by the function $p : A \to [0, 1]$ defined by

$$p(a) = \pi(\{\omega \in \Omega : \sigma(\omega) = a\}) = \sum_{\{\omega\in\Omega:\sigma(\omega)=a\}} \pi(\omega) \qquad \text{for all } a \in A, .$$

Proposition 3 Let $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be a strategic game, and let $x = (x_1, \cdots, x_n)$ be a mixed strategy Nash equilibrium of G. Then, there is a correlated equilibrium $\left\langle(\Omega, \pi), (\mathcal{P}_i)_{i\in N}, (\sigma_i)_{i\in N}\right\rangle$ whose induced probability distribution over action profiles is the same as x's distribution.

Proof : Let $\left\langle(\Omega, \pi), (\mathcal{P}_i)_{i\in N}, (\sigma_i)_{i\in N}\right\rangle$ be defined as follows:

- $\Omega = A$
- $\pi(a) = \prod_{i\in N} x_i(a_i)$
- $\mathcal{P}_i(a) = \{b \in A : b_i = a_i\}$
- $\sigma_i(a) = a_i$

We claim that $\left\langle(\Omega, \pi), (\mathcal{P}_i)_{i\in N}, (\sigma_i)_{i\in N}\right\rangle$ is a correlated equilibrium whose probability distribution is the same as x's distribution. Let $i \in N$. Since x is a mixed strategy Nash equilibrium, we know by Lemma 1 that for all $a_i \in A_i$

$$\begin{aligned}
\text{if } x_i(a_i) &> 0 \text{ then } U_i(x) = U_i(a_i, x_{-i}) \\
\text{if } x_i(a_i) &= 0 \text{ then } U_i(x) \geq U_i(a_i, x_{-i}).
\end{aligned}$$

Consequently, for all $a_i \in A_i$

$$x_i(a_i)U_i(a_i, x_{-i}) \geq x_i(a_i)U_i(b_i, x_{-i}) \text{ for all } b_i \in A_i. \tag{7}$$

Now let $\tau_i : A \to A_i$ be a function that is measurable with respect to $\mathcal{P}_i$. Let $a_{-i} \in A_{-i}$ be a fixed profile of actions for players other than i. Letting $b_i = \tau_i(a_i, a_{-i})$, equation (7) implies that

$$x_i(a_i)U_i(a_i, x_{-i}) \geq x_i(a_i)U_i(\tau_i(a_i, a_{-i}), x_{-i}) \text{ for all } a_i \in A_i.$$

Adding over all $a_i \in A_i$,

$$\sum_{a_i \in A_i} x_i(a_i)U_i(a_i, x_{-i}) \geq \sum_{a_i \in A_i} x_i(a_i)U_i(\tau_i(a_i, a_{-i}), x_{-i}).$$

Taking into account the definition of $U_i(a_i, x_{-i})$ and $U_i(\tau_i(a), x_{-i})$, and using the measurability of τ_i with respect to $\mathcal{P}_i$, we get

$$\begin{array}{rcl}
\sum_{a_i \in A_i} x_i(a_i) \sum_{a_{-i} \in A_{-i}} (\prod_{j \in N \setminus \{i\}} x_j(a_j))u_i(a_i, a_{-i}) & \geq & \sum_{a_i \in A_i} x_i(a_i) \sum_{a_{-i} \in A_{-i}} (\prod_{j \in N \setminus \{i\}} x_j(a_j))u_i(\tau_i(a), a_{-i}) \\
\sum_{a \in A}(\prod_{j \in N} x_j(a_j))u_i(a_i, a_{-i}) & \geq & \sum_{a \in A}(\prod_{j \in N} x_j(a_j))u_i(\tau_i(a), a_{-i}) \\
\sum_{a \in A} \pi(a)u_i(a_i, a_{-i}) & \geq & \sum_{a \in A} \pi(a)u_i(\tau_i(a), a_{-i}) \\
\sum_{a \in A} \pi(a)u_i(\sigma_i(a), \sigma_{-i}(a)) & \geq & \sum_{a \in A} \pi(a)u_i(\tau_{-i}(a), \sigma_{-i}(a)).
\end{array}$$

This shows that $\langle (\Omega, \pi), (\mathcal{P}_i)_{i \in N}, (\sigma_i)_{i \in N} \rangle$ is a correlated equilibrium of G. Its induced probability distribution over action profiles is

$$\begin{array}{rcl}
p(a) & = & \pi(\{b \in A : \sigma(b) = a\}) \\
& = & \pi(\{b \in A : b = a\}) \\
& = & \pi(a) \\
& = & \prod_{i \in N} x_i(a_i).
\end{array}$$

$\square$

Although a correlated strategy profile consists of a randomizing device used by the players, it turns out that the only feature of the device that determines whether or not the correlated strategy profile constitutes a correlated equilibrium is its induced probability distribution over the action profiles. This is shown by the next proposition.

Proposition 4 Let $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be a finite strategic game. Every correlated equilibrium probability distribution over action profiles can be obtained in a correlated equilibrium of G in which

- $\Omega = A$
- $\mathcal{P}_i(a) = \{b \in A : b_i = a_i\}$.

Proof: Let $\langle(\Omega', \pi'), (\mathcal{P}'_i, \sigma'_i)_{i\in N}\rangle$ be a correlated equilibrium of G. Consider the correlated strategy profile $\langle(\Omega, \pi), (\mathcal{P}_i, \sigma_i)_{i\in N}\rangle$ defined by

- $\Omega = A$
- $\pi(a) = \pi'(\{\omega \in \Omega : \sigma'(\omega) = a\})$ for each $a \in A$
- $\mathcal{P}_i(a) = \{b \in A : b_i = a_i\}$ for each $i \in N$ and for each $a \in A$
- $\sigma_i(a) = a_i$ for each $i \in N$.

It is clear that this correlated strategy profile induces the required distribution over action profiles. Indeed,

$$\begin{aligned} p(a) &= \pi(\{\omega \in \Omega : \sigma(\omega) = a\}) \\ &= \pi(\{a' \in A : a' = a\}) \\ &= \pi(a) \\ &= \pi'(\{\omega \in \Omega' : \sigma'(\omega) = a\}). \end{aligned}$$

It remains to show that this profile is a correlated equilibrium. Take a function $\tau_i : A \to A_i$ that is measurable with respect to $\mathcal{P}_i$. Define $\tau'_i : \Omega' \to A_i$ by $\tau'_i(\omega) = \tau_i(\sigma'(\omega)) =$

$\tau_i(\sigma'_{-i}(\omega), \sigma'_i(\omega))$. The function τ'_i is measurable with respect to $\mathcal{P}'_i$. Indeed, if $\omega' \in \mathcal{P}'_i(\omega)$, then $\sigma'_i(\omega') = \sigma'_i(\omega)$ by measurability of σ'_i with respect to $\mathcal{P}'_i$. Therefore, by definition of $\mathcal{P}_i$, $\mathcal{P}_i(\sigma'(\omega')) = \mathcal{P}_i(\sigma'(\omega))$, and both $\sigma'_i(\omega')$ and $\sigma'_i(\omega)$ belong to the same element of $\mathcal{P}_i$. Since τ_i is measurable with respect to $\mathcal{P}_i$, we conclude that $\tau'_i(\omega') = \tau_i(\sigma'_i(\omega')) = \tau_i(\sigma'_i(\omega)) = \tau'_i(\omega)$.

Also,

$$\begin{aligned}
\sum_{\omega\in\Omega} \pi(\omega) u_i(\sigma_{-i}(\omega), \tau_i(\omega)) &= \sum_{a\in A} \pi(a) u_i(a_{-i}, \tau_i(a)) \\
&= \sum_{a\in A} \sum_{\{\omega\in\Omega':\sigma'(\omega)=a\}} \pi'(\omega) u_i(\sigma'_{-i}(\omega), \tau_i(\sigma'(\omega))) \\
&= \sum_{a\in A} \sum_{\{\omega\in\Omega':\sigma'(\omega)=a\}} \pi'(\omega) u_i(\sigma'_{-i}(\omega), \tau'_i(\omega)) \\
&= \sum_{\omega\in\Omega'} \pi'(\omega) u_i(\sigma'_{-i}(\omega), \tau'_i(\omega)).
\end{aligned}$$

In particular, for $\tau_i = \sigma_i$,

$$\sum_{\omega\in\Omega} \pi(\omega) u_i(\sigma_{-i}(\omega), \sigma_i(\omega)) = \sum_{\omega\in\Omega'} \pi'(\omega) u_i(\sigma'_{-i}(\omega), \sigma'_i(\omega)).$$

Since $\langle(\Omega', \pi'), (\mathcal{P}'_i, \sigma'_i)_{i\in N}\rangle$ is a correlated equilibrium,

$$\sum_{\omega\in\Omega'} \pi'(\omega) u_i(\sigma'_{-i}(\omega), \sigma'_i(\omega)) \geq \sum_{\omega\in\Omega'} \pi'(\omega) u_i(\sigma'_{-i}(\omega), \tau'_i(\omega))$$

and therefore

$$\sum_{\omega\in\Omega} \pi(\omega) u_i(\sigma_{-i}(\omega), \sigma_i(\omega)) \geq \sum_{\omega\in\Omega} \pi(\omega) u_i(\sigma_{-i}(\omega), \tau_i(\omega)).$$

□

8 Rationality, Correlated Equilibrium and Equilibrium in Beliefs

As mentioned earlier, Nash equilibrium and correlated equilibrium are two examples of what is known as solution concepts. Solution concepts assign to each game a pattern of behavior for the players in the game. The interpretation of these patterns of behavior is not always explicit, but it is fair to say that they are usually interpreted either as descriptions of what rational people do, or as prescriptions of what rational people should do. There is a growing literature that tries to connect various game theoretic solution concepts to the idea of rationality. Rationality is generally understood as the characteristic of a player who chooses an action that maximizes his preferences, given his information about the environment in which he acts. Part of the information a player has is represented by his beliefs about the behavior of other players, their beliefs about the behavior of other players, and so on. So when one speaks of the rationality of players, one needs to take into account their *epistemic state*. There is a formal framework which is appropriate for discussing the actions, knowledge, beliefs and rationality of players. Namely, the framework of a correlated strategy profile. As defined in Section 7, a correlated strategy profile in a game G consists of

- A finite probability space (Ω, π)
- For each player $i \in N$ a partition $\mathcal{P}_i$ of Ω into events of positive probability
- For each player $i \in N$ a function $\sigma_i : \Omega \to A_i$ which is measurable with respect to $\mathcal{P}_i$.

For the present discussion we interpret a correlated strategy profile $\left\langle (\Omega, \pi), (\mathcal{P}_i)_{i\in N}, (\sigma_i)_{i\in N} \right\rangle$ as a description of the players' behavior and beliefs, as observed by an outside observer. The set Ω is the set of possible states of the world and π is the prior probability on Ω shared by all the players. For each player $i \in N$, $\mathcal{P}_i$ is a partition of Ω that represents i's information. At state $\omega \in \Omega$, player i is informed not of the state that actually occurred, but of the element $\mathcal{P}_i(\omega)$ of his partition that contains ω. Player i then uses this information and his prior π to update his beliefs about the true state of the world. Finally, the function σ_i represents the actions taken by player i at each state. In particular, $\sigma_i(\omega)$ is the action chosen by i at state ω. Although a correlated equilibrium can be interpreted as a correlated strategy profile *prescribed* by a given solution concept (that of a correlated equilibrium), here we want to interpret a correlated strategy profile as a description of what players actually do and believe. Although players cannot freely choose their beliefs (in the same way as they cannot choose their preferences), they can choose their actions. Furthermore, they have no obligation to behave according to the specified correlated strategy profile. However, ultimately players do behave in a certain way and that behavior is what is represented by the given correlated strategy profile.

Once we fix a correlated strategy profile we can address the rationality of the players. Formally,

Definition 9 Player $i \in N$ is *Bayes rational* at $\omega \in \Omega$ if his expected payoff at ω, $E(u_i(\sigma)|\mathcal{P}_i)(\omega)$, is at least as large as the amount $E(u_i(\sigma_{-i}, a_i)|\mathcal{P}_i)(\omega)$ that he would have got had he chosen action $a_i \in A_i$ instead of $\sigma_i(\omega)$.

In other words, player i is rational at a given state of the world if the action $\sigma_i(\omega)$ he chooses at that state maximizes his expected utility given his information, $\mathcal{P}_i(\omega)$, and, in particular, given his beliefs about the actions of the other players.

As before, for any finite set T, let $\Delta(T)$ be the set of all probability distributions on T. The beliefs of player i about the actions of the other players are represented by his conjectures. A conjecture of i is a probability distribution $\psi_i \in \Delta(A_{-i})$ over the elements of A_{-i}. For any $j \neq i$, the marginal of ψ_i on A_j is the conjecture of i about j induced by ψ_i. Given a correlated strategy profile $\left\langle (\Omega, \pi), (\mathcal{P}_i)_{i \in N}, (\sigma_i)_{i \in N} \right\rangle$, one can determine the conjectures that each player is entertaining at each state of the world about the actions of the other players. These conjectures are given by the following definition.

Definition 10 Given a correlated strategy profile $\langle (\Omega, \pi), (\mathcal{P}_i, \sigma_i)_{i \in N} \rangle$, the conjectures of $i \in N$ about the other players' actions are given by the function $\phi_i : \Omega \to \Delta(A_{-i})$ defined by

$$\phi_i(\omega)(a_{-i}) = \frac{\pi\left[\{\omega' \in \mathcal{P}_i(\omega) : \sigma_{-i}(\omega') = a_{-i}\}\right]}{\pi\left[\mathcal{P}_i(\omega)\right]}.$$

For each ω, $\phi_i(\omega) \in \Delta(A_{-i})$ is the conjecture of i at ω. For $j \neq i$, the marginal of $\phi_i(\omega)$ on A_j is the conjecture of i at ω about j's actions.

Given a correlated strategy profile, we can speak about what each player knows. The object of knowledge are called *events,* which are the subsets of the set of states of the world Ω. We say that player i *knows* event $E \subset \Omega$ at state ω, if $P_i(\omega) \subset E$. That is, i knows E at ω if whatever state he deems possible at ω is in E.

The next result, proved by Aumann and Brandenburger (1995), shows a remarkable relationship between the rationality of players and the concept of equilibrium in beliefs.

Theorem 2 Fix a two-person game, $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$, and let $\left\langle (\Omega, \pi), (\mathcal{P}_i)_{i\in N}, (\sigma_i)_{i\in N}\right\rangle$ be a correlated strategy profile for G. Let $\psi_1 \in \Delta(A_1)$ and $\psi_2 \in \Delta(A_2)$ be two conjectures, one about player 1's actions and the other about player 2's actions. Assume that at some state $\omega \in \Omega$ each player knows that the other is rational and that their conjectures at ω are $(\phi_1(\omega), \phi_2(\omega)) = (\psi_2, \psi_1)$. Then, (ψ_1, ψ_2) is an equilibrium in beliefs.

Proof : The fact that player i knows at ω that j's conjecture is ψ_i means that

$$\mathcal{P}_i(\omega) \subset \{\omega' \in \Omega : \phi_j(\omega')(a_i) = \psi_i(a_i) \text{ for all } a_i \in A_i\}.$$

Therefore

$$\phi_j(\omega)(a_i) = \psi_i(a_i) \text{ for all } a_i \in A_i. \tag{8}$$

Given Proposition 2 and Corollary 1, we need to show that if $\psi_i(a_i^*) > 0$, a_i^* is a best response to ψ_j, for $i, j = 1, 2$, $i \neq j$. For this purpose, assume that $\psi_i(a_i^*) > 0$ for some $a_i^* \in A_i$. Then, by definition of ϕ_j and (8), $\phi_j(\omega)(a_i^*) = \pi\left[\{\omega' \in \mathcal{P}_j(\omega) : \sigma_i(\omega') = a_i^*\}\right] > 0$. Consequently, there is $\omega' \in \mathcal{P}_j(\omega)$ such that $\sigma_i(\omega') = a_i^*$. Since player j knows at ω that player i is rational,

$$\omega' \in \mathcal{P}_j(\omega) \subset \{\omega'' \in \Omega : E\left[u_i(\sigma)|\mathcal{P}_i\right](\omega'') \geq E\left[u_i(\sigma_{-i}, a_i)|\mathcal{P}_i\right](\omega'') \text{ for all } a_i \in A_i\}.$$

Therefore,

$$E\left[u_i(\sigma)|\mathcal{P}_i\right](\omega') \geq E\left[u_i(\sigma_{-i}, a_i)|\mathcal{P}_i\right](\omega') \text{ for all } a_i \in A_i$$

and since $\sigma_i : \Omega \to A_i$ is measurable with respect to $\mathcal{P}_i$, $\sigma_i(\omega') = a_i^*$ is the action that player i chooses at all states in $\mathcal{P}_i(\omega')$. Then we can write

$$E\left[u_i(\sigma_{-i}, a_i^*)|\mathcal{P}_i\right](\omega') \geq E\left[u_i(\sigma_{-i}, a_i)|\mathcal{P}_i\right](\omega') \text{ for all } a_i \in A_i.$$

That is, for all $a_i \in A_i$

$$\begin{aligned}
\sum_{\omega'' \in \mathcal{P}_i(\omega')} \frac{\pi(\omega'')}{\pi(\mathcal{P}_i(\omega'))} u_i(\sigma_{-i}(\omega''), a_i^*) &\geq \sum_{\omega'' \in \mathcal{P}_i(\omega')} \frac{\pi(\omega'')}{\pi(\mathcal{P}_i(\omega'))} u_i(\sigma_{-i}(\omega''), a_i) \\
\sum_{a_j \in A_j} \sum_{\substack{\omega'' \in \mathcal{P}_i(\omega') \\ \sigma_j(\omega'') = a_j}} \frac{\pi(\omega'')}{\pi(\mathcal{P}_i(\omega'))} u_i(a_j, a_i^*) &\geq \sum_{a_j \in A_j} \sum_{\substack{\omega'' \in \mathcal{P}_i(\omega') \\ \sigma_j(\omega'') = a_j}} \frac{\pi(\omega'')}{\pi(\mathcal{P}_i(\omega'))} u_i(a_j, a_i) \\
\sum_{a_j \in A_j} \frac{\pi\left[\{\omega'' \in \mathcal{P}_i(\omega') : \sigma_j(\omega'') = a_j\}\right]}{\pi(\mathcal{P}_i(\omega'))} u_i(a_j, a_i^*) &\geq \sum_{a_j \in A_j} \frac{\pi\left[\{\omega'' \in \mathcal{P}_i(\omega') : \sigma_j(\omega'') = a_j\}\right]}{\pi(\mathcal{P}_i(\omega'))} u_i(a_j, a_i) \\
\sum_{a_j \in A_j} \phi_i(\omega')(a_j) u_i(a_j, a_i^*) &\geq \sum_{a_j \in A_j} \phi_i(\omega')(a_j) u_i(a_j, a_i). \qquad (9)
\end{aligned}$$

Since $\omega' \in \mathcal{P}_j(\omega)$ and player j knows at ω that i's conjecture is ψ_j, then

$$\omega' \in \mathcal{P}_j(\omega) \subset \left\{\omega'' \in \Omega : \phi_i(\omega'')(a_j) = \psi_j(a_j) \text{ for all } a_j \in A_j\right\}.$$

Therefore $\phi_i(\omega')(a_j) = \psi_j(a_j)$ for all $a_j \in A_j$, or $\phi_i(\omega') = \psi_j$. That is, i's conjecture at ω' about j's actions is ψ_j. Consequently, substituting in (9),

$$\sum_{a_j \in A_j} \psi_j(a_j) u_i(a_j, a_i^*) \geq \sum_{a_j \in A_j} \psi_j(a_j) u_i(a_j, a_i) \qquad \forall a_i \in A_i.$$

That is, a_i^* is a best response to player i's beliefs about j's actions. $\square$

The only assumptions required by Theorem 2 is that players know they are rational, and that they know each other's conjectures. In a correlated strategy profile for a two-player game, there is only one player entertaining a conjecture about the actions of player

1, namely, player 2. Similarly, player 1 is the only one who entertains a conjecture about the actions of player 2. In an n-person game, with $n > 2$, for each player, there is more than one player entertaining a conjecture about his actions. Therefore, since an equilibrium in beliefs consists of a profile of beliefs, each of which is shared by $n-1$ players, a generalization of Theorem 2 would require the players' beliefs about player i's actions, for $i \in N$, to be identical. In order to obtain these common beliefs it is not sufficient to assume that players know each other's conjectures. One need to strengthen this assumption. Also, in an equilibrium in beliefs, the common belief about player i's actions assigns positive probability only to best responses to i's conjectures about the choices of the other players. Furthermore, i's conjectures about the other players' choices is the product of his beliefs about each of the other players. In other words, an equilibrium in beliefs implicitly assumes that players believe that the other players' choices are independent. Aumann and Brandenburger (1995) show that one way to obtain common conjectures and, simultaneously, that players believe that the other players act independently, is to assume that players' conjectures are commonly known. This surprising and deep result is stated in the next theorem.

Theorem 3 Let $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be a strategic game, and let $\left\langle (\Omega, \pi), (\mathcal{P}_i)_{i\in N}, (\sigma_i)_{i\in N}\right\rangle$ be a correlated strategy profile for G. Also let $(\psi_i)_{i\in N} \in \times_{i\in N}\Delta(A_{-i})$ be a profile of conjectures, one for each player. Assume that at some state $\omega \in \Omega$ each player knows that the others are rational. Further, assume that at ω their conjectures are commonly known to be $(\psi_i)_{i\in N}$. Then, for each j, all the conjectures ψ_i of players i other than j, induce the same belief $\varphi_j \in \Delta(A_j)$ about j's actions, and the resulting profile of beliefs, $(\varphi_i)_{i\in N}$, is an equilibrium in beliefs.

8.1 Rationality and Correlated Equilibrium

The previous result shows a surprising relationship between the players' rationality and the concept of equilibrium in beliefs. If at some state of the world players know that everybody is rational, and if their conjectures are commonly known at that state, then their beliefs about each player's actions are in equilibrium. It is not that their actions constitute an equilibrium, but that their beliefs do. The question that naturally arises is: are there any epistemic conditions on the players that would induce them to play according to equilibrium? To answer this we turn to Aumann (1987), where it is stated that if players are rational at every state, then their behavior constitutes a correlated equilibrium. Therefore, in order to obtain an equilibrium behavior, a sufficient condition is not that players be rational, or that they know that they are rational at some particular state, but that their rationality be common knowledge. And if it is common knowledge that all players are rational, then their behavior is not necessarily a Nash equilibrium, but a correlated equilibrium.

Theorem 4 Let G be a strategic game, and let $\left\langle (\Omega,\pi), (\mathcal{P}_i)_{i\in N}, (\sigma_i)_{i\in N}\right\rangle$ be a correlated strategy profile for G. If each player is rational at each state of the world, then $\left\langle (\Omega,\pi), (\mathcal{P}_i)_{i\in N}, (\sigma_i)_{i\in N}\right\rangle$ is a correlated equilibrium.

Proof : Let $\tau_i : \Omega \to A_i$ be a function that is measurable with respect to $\mathcal{P}_i$. Since i is Bayes rational at ω

$$E(u_i(\sigma)|\mathcal{P}_i)(\omega) \geq E(u_i(\sigma_{-i}, a_i)|\mathcal{P}_i)(\omega) \qquad \forall a_i \in A_i.$$

That is,

$$\sum_{\omega'\in\mathcal{P}_i(\omega)} \frac{\pi(\omega')}{\pi(\mathcal{P}_i(\omega))} u_i(\sigma_{-i}(\omega'), \sigma_i(\omega')) \geq \sum_{\omega'\in\mathcal{P}_i(\omega)} \frac{\pi(\omega')}{\pi(\mathcal{P}_i(\omega))} u_i(\sigma_{-i}(\omega'), a_i) \qquad \forall a_i \in A_i.$$

In particular, for $a_i = \tau(\omega) = \tau(\omega')$ for all $\omega' \in \mathcal{P}_i(\omega)$,

$$\sum_{\omega'\in\mathcal{P}_i(\omega)} \frac{\pi(\omega')}{\pi(\mathcal{P}_i(\omega))} u_i(\sigma_{-i}(\omega'), \sigma_i(\omega')) \geq \sum_{\omega'\in\mathcal{P}_i(\omega)} \frac{\pi(\omega')}{\pi(\mathcal{P}_i(\omega))} u_i(\sigma_{-i}(\omega'), \tau(\omega')).$$

Multiplying both sides by $\pi(\omega)$ and adding over all the elements of $\mathcal{P}_i$ we get

$$\sum_{\omega\in\Omega} \pi(\omega)u_i(\sigma_{-i}(\omega), \sigma_i(\omega)) \geq \sum_{\omega\in\Omega} \pi(\omega)u_i(\sigma_{-i}(\omega), \tau(\omega)).$$

□

9 Bayesian Games

Thus far, we have considered static games, which are objects of the form $\langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$. Although these games have many applications, they are not readily suitable for the analysis of situations involving asymmetric information. Indeed, an implicit assumption behind the definition of a static game is that all players have the same information about the relevant aspects of the situation. In particular, all players have the same information about the sets of actions and preferences of all players. A static game seems suitable to model strategic interactions like the prisoner's dilemma, rock scissors paper, and even chess. At the time they choose their actions, all the players have exactly the same information. There might be what is called strategic uncertainty, namely, uncertainty about what the players will do, but there is no uncertainty about the rules of the game and about the preferences of the players. But how would one translate a game of cards like bridge or poker into a static game? In a game of cards, at the time of choosing his actions, each player knows the cards he holds in his hand, but does not know the cards of his opponents. He only has a belief about the cards held by his opponents. In order to make a sound choice, a player will try to predict the actions of his opponents, but for this it is crucial to use his beliefs about the cards they hold. For the same reason, his opponents should use their beliefs about their own opponents' cards in order to make a sound choice. Thus, the beliefs about the cards held by each player should be part of a description of a game with asymmetric information. Further, in order to predict his opponents' actions, a player also needs to assess his opponents' beliefs about his own cards. This seems to induce an intractable infinite regress of beliefs, and beliefs about beliefs. Harsanyi (1967) provided the basic structure to describe and analyze strategic situations where players are asymmetrically informed. This structure is called a Bayesian game.

Definition 11 A *Bayesian Game* is a system $\langle N, (\Omega, \mu), (A_i, \mathcal{P}_i, u_i)_{i\in N}\rangle$ where

- N is the set of players
- Ω is the set of states of nature
- μ is the players' common prior belief (a probability measure over the set of states)
- A_i is player i's set of actions
- $\mathcal{P}_i$ is player i's information partition (a partition of Ω into sets of positive measure). Each element of the partition is referred to as a player's type.
- $u_i : \times_{i\in N} A_i \times \Omega$ is player i's Bernoulli utility function (a function over pairs (a, ω) where $a \in A$ and $\omega \in \Omega$, the expected value of which represents the player's preferences among lotteries over the set of such pairs).

The interpretation of a Bayesian game is as follows. The basic uncertainty is represented by the probability space (Ω, μ) of all states of nature and the prior probability over them. Each state represents a realization of all the parametric uncertainty of the model. For instance, in a game of cards, each state represents each of the possible card deals. The information of player $i \in N$ is represented by his information partition $\mathcal{P}_i$. While states in the same element of the partition cannot be distinguished by the player, he can distinguish between states that belong to different partition cells. In a game of cards, for instance, each partition cell represents a particular set of cards dealt to the player. The probability measure μ represents the players' prior belief about the state of nature. This prior belief will be used along with the information obtained by each player to form beliefs about the other players' information. The set of actions of player i is A_i. Note that there is no loss of generality in assuming that this set does not depend on the state of nature. One can always add unavailable actions and assign them intolerable disutility. Finally, u_i is the payoff function that associates to each state of nature and action profile a utility level. Note that since the state of the world is unknown to the player at the time of making his choice, a player faces a lottery for any given action profile. The assumption is that the player evaluates this lottery according to the expected value of u_i with respect to that lottery.

Let $\langle N, (\Omega, \mu), (A_i, \mathcal{P}_i, u_i)_{i \in N} \rangle$ be a Bayesian game. A strategy for player $i \in N$ is a function $\sigma_i : \Omega \to A_i$ that is measurable with respect to $\mathcal{P}_i$. We denote the set of strategies for player i by $\mathcal{B}_i$. That is, $\mathcal{B}_i = \{\sigma_i : \Omega \to A_i : \sigma_i \text{ is measurable w.r.t. } \mathcal{P}_i\}$. The interpretation of a strategy in a Bayesian game is the usual one. For each state of nature $\omega \in \Omega$, $\sigma_i(\omega)$ is the action chosen by player i at ω. The measurability requirement imposes that player i's actions depend only on his information. If player i cannot distinguish between two states of nature, then he must choose the same action at both states. Player i evaluates a profile $\sigma : \Omega \to A$ of strategies according to the expected value of u_i with respect to μ.

In order to define an equilibrium notion for Bayesian games we follow the same idea used for the definition of a mixed strategy equilibrium. Namely, we translate the Bayesian game into a standard game, and then define an equilibrium of the Bayesian game as the Nash equilibirum of the induced game.

Definition 12 A *Bayesian equilibrium* of a Bayesian game $\langle N, (\Omega, \mu), (A_i, \mathcal{P}_i, u_i)_{i \in N} \rangle$ is a Nash equilibrium of the strategic game: $\langle N, (\mathcal{B}_i)_{i \in N}, (U_i)_{i \in N} \rangle$ where for each profile $\sigma : \Omega \to A$ of strategies, $U_i(\sigma) = E_\mu[u_i(\sigma(\omega), \omega)]$ is i's expected utility with respect to μ.

A Bayesian equilibrium of a Bayesian game is a Nash equilibrium of a properly defined static game. As such, conditions for its existence can be derived from Theorem 1. However, in many situations one is interested in particular kinds of equilibria. Specifically, in the analysis of auctions or of the war of attrition, one is often interested in efficient outcomes. In a single object auction, efficient outcomes are characterized by the fact that in equilibrium the object is allocated to the buyer who values it most. According to many standard auction rules, the object goes to the highest bidder. Therefore, in such auctions, to guarantee an efficient outcome, one would need a monotone equilibrium, namely, one in which bidders bids are higher the higher their valuations for the object are. Athey (2001) shows conditions under which a Bayesian equilibrium exists where strategies are non-decreasing. The crucial conditions are that the players' types can be represented by a one-dimensional variable, and that, fixing a nondecreasing strategy for each of a player's opponents, this player's expected payoffs satisfies a single-crossing property. This single-crossing property roughly says that if a high action is preferred to a low action for a given type t, then the same must be true for all types higher than t. McAdams (2003) extended Athey's result to the case where types and actions are multidimensional and partially ordered.

9.1 The asymmetric information version of the war of attrition

We have seen that, when applied to the war of attrition, as modeled by a standard strategic game or by its mixed extension, the notion of Nash equilibrium does not yield a satisfactory prediction.[1] In the former case all the equilibria involve no fight, and in the latter case the equilibrium dictates a more aggressive behavior to the player who values the contested object less. In what follows, we analyze the war of attrition as a Bayesian game. That is, we assume that the players are ex-ante symmetric but they have private information about their value for the contested object.

A Bayesian game that represents the war of attrition is given by $\langle N, \Omega, (A_i, \mu_i, \mathcal{P}_i, u_i)_{i \in N} \rangle$ where

[1]The war of attrition was analyzed in Maynard Smith (1974). For an analysis of the asymmetric information version of the war of attrition, see Krishna and Morgan (1997).

- $N = \{1, 2\}$
- $\Omega = [0, \infty)^2 = \{(v_1, v_2) : 0 \leq v_i < \infty,\ i = 1, 2\}$
- $A_i = [0, \infty)$ for $i = 1, 2$
- $\mathcal{P}_i(\hat{v}_1, \hat{v}_2) = \{(v_1, v_2) \in \Omega : v_i = \hat{v}_i\}$ for $i = 1, 2$
- $\mu((v_1, v_2) \leq (\hat{v}_1, \hat{v}_2)) = F(\hat{v}_1) \times F(\hat{v}_2)$
- $u_i((a_1, a_2), (v_1, v_2)) = \begin{cases} -a_i & \text{if } a_i \leq a_j \\ v_i - a_j & \text{if } a_i > a_j \end{cases}$

Here the set of types of player i, for $i = 1, 2$, is represented by the player's willingness to fight, v_i. The players' willingness to fight are drawn independently from the same distribution F. A state of the world is, therefore, a realization (v_1, v_2) of the players' types, and at that state, each player is informed only of his type. Finally, the utility of a player is his valuation for the prey, if he obtains it, net of the time spent fighting for it. We are interested in a symmetric equilibrium in which both players use a symmetric, strictly increasing strategy $\beta : [0, \infty) \to [0, \infty)$, where $\beta(v)$ is the time at which a player with willingness to fight v is dictated by the equilibrium to give up. Such an equilibrium would imply that types who value the prey more, are willing to fight more. Further, the probability of observing a fight in equilibrium would not be 0 (in fact, it would be 1.)

It turns out that a symmetric equilibrium strategy is given by

$$\beta(v) = \int_0^v \frac{x f(x)}{1 - F(x)} dx,$$

where f denotes the derivative of F. To see this, assume that player j behaves according to β and that player i chooses to give up at t. Letting z be the type such $\beta(z) = t$, the expected utility of player i from choosing t is

$$U(v_i, z) = \int_0^z (v_i - \beta(y)) f(y) dy - \beta(z)(1 - F(z)).$$

Taking derivatives with respect to z, and using the fact that $\beta'(z) = zf(z)/[1 - F(z)]$ we obtain

$$\begin{aligned}\frac{\partial U}{\partial z}(v_i, z) &= v_i f(z) - \beta'(z)(1 - F(z)) \\ &= (v_i - z)f(z),\end{aligned}$$

which is positive for $z < v_i$, and negative for $z > v_i$. As a result, the expected utility of player i with willingness to pay v_i is maximized at $z = v_i$, which implies that the optimal choice is $\beta(v_i)$.

Thus, modeling the war of attrition as an asymmetric game has allowed us to find an equilibrium in which players with higher willingness to fight fight more, and there is a non-negligible probability of observing a fight.

10 Evolutionary Stable Strategies

The notion of the Nash equilibrium concept involves players choosing actions that maximize their payoffs given the choices of the other players. The usual interpretation of a Nash equilibrium is as a pattern of behavior that rational players *should* adopt. However, Nash equilibria are sometimes interpreted more descriptively as patterns of behavior that rational players *do* adopt. Certainly, rationality of players is neither a necessary condition nor a sufficient one for players to play a Nash equilibrium. The relationship between rationality and the various solution concepts is not apparent and has been the focus of an extensive literature (see, for example, Aumann and Brandenburger (1995), Aumann (1995) Aumann (1987), Brandenburger and Dekel (1987)). Nonetheless, the notion of a Nash equilibrium evokes the idea of players consciously making choices with the deliberate objective of maximizing their payoffs. It is therefore quite remarkable that a concept almost identical to that of Nash equilibrium has emerged from the biology literature. This concept describes a population equilibrium where unconscious organisms are programmed to choose actions with no deliberate aim. In this equilibrium, members of the population meet at random over and over again to interact. At each interaction, these players act in a pre-programmed way and the result of their actions is a gain in biological fitness. Fitness is a concept related to the reproductive value or survival capacity of an organism. In a temporary equilibrium, the fitness gains are such that the proportions of individuals that choose each one of the possible actions remain constant. However, this temporary equilibrium may be disturbed by the appearance of a mutation, which is a new kind of behavior. This mutation may upset the temporary equilibrium if its fitness gains are such that the new behavior spreads over the population. Alternatively, if the fitness gains of the original population outweigh those of the mutation, then the new behavior will fail to propagate and will eventually disappear. In a population equilibrium, the interaction of any mutant with the whole population awards the mutant insufficient fitness gains, and as a result the mutants disappear. The notion of a population equilibrium is formalized by means of the concept of an evolutionary stable strategy, introduced by Maynard Smith and Price (1973).

In what follows we restrict our attention to symmetric two-player games. So let $G = \langle \{1,2\}, \{A_1, A_2\}, \{u_1, u_2\} \rangle$ be a game such that $A_1 = A_2 = \overline{A}$, and such that for all $a, b \in \overline{A}$, $u_1(a,b) = u_2(b,a)$. An evolutionary stable strategy is an action in $\overline{A}$ such that if all members of the population were to choose that action, no sufficiently small proportion of mutants choosing an alternative action would succeed in invading the population. Alternatively, an evolutionary stable strategy is an action in $\overline{A}$ such that if all the members of the population were to choose that action, the population would reject all sufficiently small mutations involving a different action.

More specifically, suppose that all members of the population are programmed to choose $a \in \overline{A}$, and then a proportion ε of the population mutates and adopts action $b \in \overline{A}$. In that case, the probability that a given member of the population meets a mutant is ε, while the probability of meeting a member that plays a is $1 - \varepsilon$. Therefore, the mutation will not propagate and will vanish if the expected payoff of a mutant is less than the expected payoff of a member of the majority. Otherwise it will propagate. This leads to the following definition.

Definition 13 An action $a \in \overline{A}$ is an evolutionary stable strategy of G if there is an $\bar{\varepsilon} \in (0,1)$ such that for all $\varepsilon \in (0, \bar{\varepsilon})$, and for all $b \in \overline{A}$

$$(1-\varepsilon)u_1(a,a) + \varepsilon u_1(a,b) > (1-\varepsilon)u_1(b,a) + \varepsilon u_1(b,b). \tag{10}$$

The following result shows that the concept of an evolutionary stable strategy is very close to the notion of a Nash equilibrium.

Proposition 5 If $a \in \overline{A}$ is an evolutionary stable strategy of G, then (a,a) is a Nash equilibrium. And if (a,a) is a strict Nash equilibrium then a is an evolutionary stable strategy.

Proof : If $u_1(a,a) > u_1(b,a)$ for all $b \in \overline{A} \setminus \{b\}$, then inequality (10) holds for all sufficiently small $\varepsilon > 0$. If $u_1(b,a) > u_1(a,a)$ for some $b \in \overline{A}$, the reverse inequality holds for all sufficiently small ε. □

11 Future Directions

Static games have been shown to be a useful framework for analyzing and understanding many situations that involve strategic interaction. At present, a large body of literature is available that develops various solution concepts, some of which are refinements of Nash equilibrium and some of which are coarsenings of it. Nonetheless, several areas for future research remain. One is the application of the theory to particular games to better understand the situations they model, for example auctions. In many markets trade is conducted by auctions of one kind or another, including markets for small domestic products as well as some centralized electricity markets where generators and distributors buy and sell electric power on a daily basis. Also, auctions are used to allocate large amounts of valuable spectrum among telecommunication companies. It would be interesting to calculate the equilibria of many real life auctions. Simultaneously, future research should also focus on the design of auctions whose equilibria have certain desirable properties.

Another future direction would be to empirically and experimentally test the theory. The various equilibrium concepts predict certain kinds of behavior in certain games. Our confidence in the predictive and explanatory power of the theory depends on its performance in the field and in the laboratory. Moreover, the experimental and empirical results should provide valuable feedback for further development of the theory. Although some valuable experimental and empirical tests have already been performed (see McKelvey and Palfrey (1992), O'Neill (1987), Walker and Wooders (2001), and Palacios-Huerta (2003) to name a few), the empirical aspect of game theory in general, and of static games in particular, remains underdeveloped.

References

Athey, S. (2001). Single crossing properties and the existence of pure strategy equilibria in games of incomplete information. *Econometrica 69*, 861–889.

Aumann, R. J. (1974). Subjectivity and correlation in randomized strategies. *Journal of Mathematical Economics 1*, 67–96.

Aumann, R. J. (1987). Correlated equilibrium as an expression of bayesian rationality. *Econometrica 55*, 1–18.

Aumann, R. J. (1995). Backward induction and common knowledge of rationality. *Games and Economic Behavior 8*, 6–19.

Aumann, R. J. and A. Brandenburger (1995). Epistemic conditions for Nash equilibrium. *Econometrica 63*, 1161–1180.

Binmore, K. (2007). *Playing for Real.* New York: Oxford University Press.

Brandenburger, A. and E. Dekel (1987). Rationalizability and correlated equilibria. *Econometrica 55*, 1391–1402.

Crawford, V. (1990). Equilibrium without independence. *Journal of Economic Theory 50*, 127–154.

Fudenberg, D. and J. Tirole (1991). *Game Theory.* Cambridge, MA: MIT Press.

Geanakoplos, J. (2003). Nash and Walras equilibrium via Brouwer. *Economic Theory 21*, 585–603.

Glicksberg, I. L. (1952). A further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points. *Proceedings of the American Mathematical Society 3*, 170–174.

Harsanyi, J. (1967). Games with incomplete information played by 'Bayesian' players, Parts i, ii and iii. *Management Science 14*, 159–82, 320–34, 486–502.

Krishna, V. and J. Morgan (1997). An analysis of the war of attrition and the all-pay auction. *Journal of Economic Theory 72*, 343–362.

Maynard Smith, J. (1974). The theory of games and the evolution of animal conflicts. *Journal of Theoretical Biology 47*, 209–221.

Maynard Smith, J. and G. R. Price (1973). The logic of animal conflict. *Nature 246*, 15–18.

McAdams, D. (2003). Isotone equilibrium in games of incomplete information. *Econometrica 71*, 1191–1214.

McKelvey, R. D. and T. R. Palfrey (1992). An experimental study of the centipede game. *Econometrica 60*, 803–836.

Nash, J. F. (1950). Equilibrium points in n-person games. *Proceedings of the National Academy of Sciences of the United States of America 36*, 48–49.

O'Neill, B. (1987). Nonmetric test of the minimax theory of two-person zero-sum games. *Proceedings of the National Academy of Sciences 84*, 2106–2109.

Osborne, M. J. (2004). *An Introduction to Game Theory*. New York: Oxford University Press.

Osborne, M. J. and A. Rubinstein (1994). *A Course in Game Theory*. Cambridge: MIT Press.

Palacios-Huerta, I. (2003). Professionals play minimax. *Review of Economic Studies 70*, 395–415.

von Neumann, J. (1928). Zur theorie der gesellschapfsspiele. *Mathematische Annalen 100*, 295–320.

von Neumann, J. (1959). On the theory of games of strategy. In A. W. Tucker and R. D. Luce (Eds.), *Contributions to the Theory of Games*, Volume IV. Princeton, NJ: Princeton University Press.

Walker, M. and J. Wooders (2001). Minimax play at wimbledon. *American Economic Review 91*, 1521–1538.

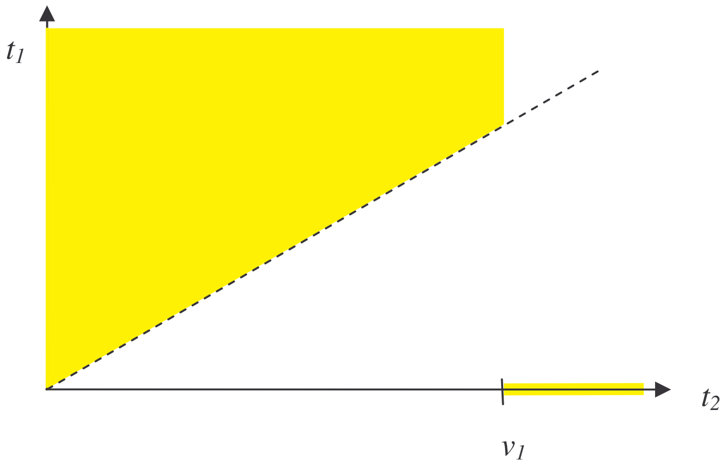

Figure 1: Player 1's best response correspondence

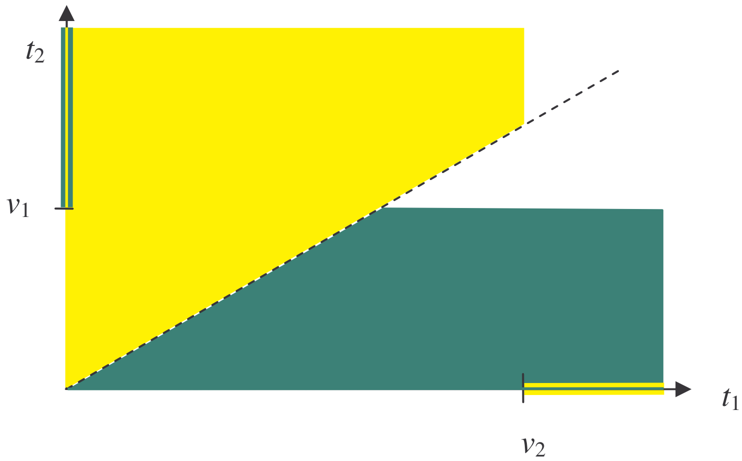

Figure 2: The equilibria