An alternative proof of Hardy Littlewood and Pólya (1929) necessary condition for majorization*

Oscar Volij†

February 27, 2016

One of the fundamental mathematical results in inequality measurement, due to Hardy Littlwood and Pólya [2], states that a necessary and sufficient condition for a vector $y \in \mathbb{R}^n_+$ to majorize another vector $x \in \mathbb{R}^n_+$ is the existence of a doubly stochastic matrix Q such that $x = y^T Q$. The standard proof of the necessity of this condition is elementary but somewhat indirect. It first shows that when y majorizes x it is possible to move from y to x by a finite sequence of non-regressive transfers, and then notices that each one of these transfers can be expressed by means of a simple doubly stochastic matrix. The desired doubly stochastic matrix is then the product of these simple matrices.

In this note we offer a direct proof, based on the minimax theorem for zero sum games. The idea of the proof is not new. It resembles the one used by Blackwell [1] in his beautiful characterization of the *at least as informative* relation on experiments.

Vectors are always $n \times 1$ matrices, namely columns. The inner product of two vectors x, y is written $x \cdot y$. For any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n_+$ let $x_{(1)} \leq \cdots \leq x_{(n)}$ denote the components of x in non-decreasing order.

For any $x, y \in \mathbb{R}^n_+$ we say that x is *majorized* by y, denoted $x \preccurlyeq y$ if

$$\begin{aligned} \sum_{i=1}^{k} x_{(i)} &\geq \sum_{i=1}^{k} y_{(i)} \qquad k = 1, \ldots, n-1 \\ \sum_{i=1}^{n} x_{(i)} &= \sum_{i=1}^{n} y_{(i)} \end{aligned}$$

Or, equivalently, if

$$\begin{aligned} \sum_{i=k}^{n} x_{(i)} &\leq \sum_{i=k}^{n} y_{(i)} \qquad k = 2, \ldots, n \\ \sum_{i=1}^{n} x_{(i)} &= \sum_{i=1}^{n} y_{(i)} \end{aligned}$$

*I thank Casilda Lasso de la Vega for helpful comments.
†Ben-GurionUniversity of the Negev

We first prove the following preliminary result.

Lemma 1 Let $x, y \in \mathbb{R}^n_+$ be two vectors such that x is majorized by y. Let $v \in \mathbb{R}^n_+$. Then, there is a permutation matrix P such that $x \cdot v \leq y^T P v$.

Proof : Let P_x be a permutation matrix that orders the components of x comonotonically with v. That is, letting $\widehat{x} = x^T P_x$, we have that $(\widehat{x}_i - \widehat{x}_j)(v_i - v_j) \geq 0$ for $i, j = 1, \ldots, n$. Then, it can be checked that

$$x \cdot v \leq \widehat{x} \cdot v = \sum_{i=1}^{n} \widehat{x}_i v_i = \sum_{i=1}^{n} x_{(i)} v_{(i)}.$$

Similarly, let P_y be a permutation matrix that orders the components of y comonotonically with v. That is, letting $\widehat{y} = y^T P_y$, $(\widehat{y}_i - \widehat{y}_j)(v_i - v_j) \geq 0$ for $i, j = 1, \ldots, n$ and hence $\widehat{y} \cdot v = \sum_{i=1}^{n} y_{(i)} v_{(i)}$. Then,

$$\begin{aligned}
x \cdot v &\leq \sum_{i=1}^{n} x_{(i)} v_{(i)} \\
&= v_{(1)} \sum_{i=1}^{n} x_{(i)} + (v_{(2)} - v_{(1)}) \sum_{i=2}^{n} x_{(i)} + \cdots + (v_{(n)} - v_{(n-1)}) x_{(n)} \\
&\leq v_{(1)} \sum_{i=1}^{n} y_{(i)} + (v_{(2)} - v_{(1)}) \sum_{i=2}^{n} y_{(i)} + \cdots + (v_{(n)} - v_{(n-1)}) y_{(n)} \\
&= \sum_{i=1}^{n} y_{(i)} v_{(i)} \\
&= \widehat{y} \cdot v \\
&= y^T P_y v
\end{aligned}$$

where the third line follows from the fact that x is majorized by y. The permutation matrix P_y is the one we are looking for. □

We can now proove the following.

Proposition 1 (Hardy, Littlewood and Pólya) Let $x, y \in \mathbb{R}^n_+$ be two vectors. There is a doubly stochastic matrix Q such that $x = y^T Q$ if and only if $x \preccurlyeq y$.

Proof : For the only if part see Theorem A.2.4 in Marshall and Olkin [3]. For the if part let $x, y \in \mathbb{R}^n_+$ and assume that $x \preccurlyeq y$. Consider the following two-person zero sum game. Player 1 chooses an $n \times n$ doubly stochastic matrix, and player 2 chooses a vector $v \in [0,1]^n$. Denote by $\mathcal{V}$ the set of all such vectors and by $\mathcal{M}$ the set of $n \times n$ doubly stochastic matrices. The payoff function for player 1 is defined by

$$h(M, v) = (y^T M - x) \cdot v$$

The sets $\mathcal{V}$ and $\mathcal{M}$ are compact and convex. Additionally, h is linear in each of its arguments. Therefore, by the Nash equilibrium existence theorem (see [4], Proposition 20.3), there is a doubly stochastic matrix M_0 and a vector v_0 such that

$$h(M, v_0) \leq h(M_0, v_0) \leq h(M_0, v) \qquad \forall M \in \mathcal{M}, \forall v \in \mathcal{V} \tag{1}$$

By Lemma 1, there is a permutation matrix P such that

$$h(P, v_0) = (y^T P - x) \cdot v_0 \geq 0$$

Since permutation matrices are doubly stochastic, it follows from (1) that

$$0 \leq h(M_0, v) \qquad \forall v \in \mathcal{V}$$

or

$$0 \leq (y^T M_0 - x) \cdot v \qquad \forall v \in \mathcal{V}$$

Choosing $v = (0, \ldots, 0, 1, 0, \ldots, 0)$ we obtain that the ith component of $(y^T M_0 - x)$ satisfies $(y^T M_0 - x)_i \geq 0$. Since

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} (y^T M_0)_i$$

we obtain $\sum_{i=1}^{n} (y^T M_0 - x)_i = 0$. Hence $(y^T M_0 - x)_i = 0$ for $i = 1, \ldots, n$. In other words, $x = y^T M_0$ and thus M_0 is the doubly stochastic matrix that we are looking for. □

References

[1] Blackwell, D. Equivalent comparisons of experiments. *The annals of mathematical statistics 24*, 2 (1953), 265–272.

[2] Hardy, G. H., Littlewood, J. E., and Pólya, G. Some simple inequalities satisfied by convex functions. *Messenger Math 58*, 145-152 (1929), 310.

[3] Marshall, A. W., Olkin, I., and Arnold, B. C. Inequalities: Theory of majorization and its applications. *Springer Series in Statistics* (2011).

[4] Osborne, M. J., and Rubinstein, A. *A course in game theory.* MIT press, 1994.