Condorcet Winners and Social Acceptability*

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Abstract

We say that an alternative is socially acceptable if the number of individuals who rank it among their most preferred half of the alternatives is at least as large as the number of individuals who rank it among the least preferred half. A Condorcet winner may not necessarily be socially acceptable. However, if preferences are single-peaked, single-dipped, or satisfy the single-crossing property, any Condorcet winner is socially acceptable. We identify maximal families of preferences that guarantee that Condorcet winners are socially acceptable.

Keywords: Condorcet winner, single-peaked preferences, single-crossing.

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1 Introduction

An alternative is a Condorcet winner if there is no other alternative that is preferred to it by more than half of the individuals. More generally, for $q \geq 1/2$, an alternative

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is a q-Condorcet winner if there is no alternative that is preferred to it by more than a proportion $1 - q$ of the individuals. On the other hand, an alternative is socially acceptable if it is ranked among the top half of the alternatives by as many individuals as those who rank it among the bottom half of the alternatives. Failing to be socially acceptable can be regarded as a drawback since a majority of voters may be uncomfortable with such an alternative. It turns out that unless q is high enough, a q-Condorcet winner may not be socially acceptable. We show, however, that if preferences are single-peaked, single-dipped, or if they satisfy the single-crossing property, any Condorcet winner is socially acceptable. A family of preferences that, like those just mentioned, guarantees that for any profile taken from it all its Condorcet winners are socially acceptable, is said to be regular. We also identify large families of preferences which are not only regular but also maximal with respect to this property.

The paper is organized as follows. After introducing some basic definitions, Section 2 shows that a Condorcet winner may fail to be socially acceptable. Section 3 shows that Condorcet winners of single-peaked preference profiles are guaranteed to be socially acceptable. Section 4 shows that Condorcet winners of preference profiles with the single-crossing property are also guaranteed to be socially acceptable. Section 5 discusses other preference domains and shows that, while the family of single-dipped preferences is regular, the class of group-separable preferences is not. Lastly, Section 6 identifies large families of preferences that are maximal with respect to the regularity property.

2 Definitions

Let $A = \{a_1, \ldots, a_K\}$ be a set of K alternatives and let $N = \{1, \ldots, n\}$ be a set of individuals. Also, let $\mathcal{P}$ be the subset of complete, transitive and antisymmetric binary relations on A. We will refer to the elements of $\mathcal{P}$ as preference relations or simply as preferences. A *preference profile* is a mapping $\pi = (\succ_1, \ldots, \succ_n)$ of preference relations on A to the individuals in N. For any subset $C \subseteq \mathcal{P}$ of preference relations, $\mu_\pi(C) = |\{i \in N : \succ_i \in C\}|$ is the number of individuals whose preference relations are in C. Also, $\pi(N) = \{\succ \in \mathcal{P} : \exists i \in N \text{ s.t. } \succ_i = \succ\}$ is the set of preferences that are present in the profile π. For any two alternatives $a, a' \in A$, $C(a \to a') = \{\succ \in \mathcal{P} : a \succ a'\}$ denotes the set of preference relations according to which a is preferred to a'.

Definition 1 Let π be a preference profile, and let $a \in A$ be an alternative. We say that a is a *Condorcet winner* for π if for every alternative $a' \in A$ the number of individuals who prefer a to a' is at least as large as the number of individuals who prefer a' to a.

For any preference relation $\succ$ and for any alternative $a \in A$, the rank of a in $\succ$, denoted by $\text{rank}_\succ(a)$, is 1 + the number of alternatives that are strictly preferred to a according to $\succ$. Formally, $\text{rank}_\succ(a) = K - |\{a' \in A \setminus \{a\} : a \succ a'\}|$. Alternatives whose ranks in $\succ$ are less than $(K+1)/2$ are said to be placed *above the line* by $\succ$, those whose ranks are greater than $(K+1)/2$ are said to be placed *below the line* by $\succ$, and those whose ranks are exactly $(K+1)/2$ are said to be placed *on the line* by $\succ$. For instance, if $K = 5$ and a voter's preference relation is given by $a_1 \succ a_2 \succ a_3 \succ a_4 \succ a_5$, he places alternatives a_1 and a_2 above the line, alternatives a_4 and a_5 below the line, and alternative a_3 on the line.

We now define the concept of social acceptability.

Definition 2 Let π be a preference profile. We say that alternative $a \in A$ is *socially acceptable for* π if the number of individuals whose preferences place a above the line is at least as large as the number of individuals whose preferences place it below the line.

Mahajne and Volij [7] showed that every preference profile has a socially acceptable alternative. In this paper we are interested in preference profiles whose Condorcet winners are socially acceptable. We call such profiles, *regular* preference profiles. Note that preference profiles with no Condorcet winners are regular.

If $K = 3$, all preference profiles are regular. To see this, let $A = \{a, b, c\}$ and assume that alternative a is not socially acceptable for some preference profile π. Then, the number of voters in π that rank a last in their preference relations is bigger than the number of voters that rank a at the top of their preference relations. Consider the remaining voters in the profile, namely those who rank a in the second place. Either for at least half of them b is preferred to a, or for at least half of them c is preferred to a. In either case, a is not a Condorcet winner.

As the following example shows, when the number of alternatives is at least four, a Condorcet winner may not be socially acceptable.

Example 1 Assume $A = \{a, b, c, d\}$ and consider the following preference profile[1]: $(abcd, acbd, cdab, cbad, bdac)$. It can be seen that whereas alternative a is a Condorcet winner, it is not socially acceptable. The only socially acceptable alternatives are b and c.

The concept of a Condorcet winner can be strengthened by requiring that the alternative be preferred to any other alternative by at least a given proportion of the voters.

Definition 3 Let $\pi = (\succ_1, \ldots, \succ_n)$ be a preference profile, and let $a \in A$ be an alternative. For $q \in (1/2, 1]$, we say that a is a q-*Condorcet winner* for π if for every alternative $a' \in A$ the number of individuals who prefer a to a' is greater or equal to a fraction q of the number of individuals – namely, if $\mu_\pi(C(a \to a')) \geq qn$ for all $a' \in A \setminus \{a\}$.

The concepts of q-Condorcet winner and q-majority equilibrium have been studied in Greenberg [5], Saari [11], Baharad and Nitzan [1], and Courtin et al. [3]. The

[1] We adopt the convention of denoting the preference relation $a \succ b \succ c \succ d$ by $abcd$.

following result shows that for large enough q, any q-Condorcet winner is socially acceptable. The specific bound, however, depends on the number K of alternatives. Specifically, letting

$$\overline{q}_K = \begin{cases} \frac{3K-4}{4K-4} & \text{if } K \text{ is even} \\ \frac{3K-5}{4K-4} & \text{if } K \text{ is odd} \end{cases}$$

we have the following.

Proposition 1 Let $q \geq \overline{q}_K$ and let π be a preference profile for which alternative a is a q-Condorcet winner. Then a is socially acceptable.

Proof : Let a be a q-Condorcet winner for π. Let

$$W_\pi(a) = \sum_{a' \in A \setminus \{a\}} \mu_\pi(C(a \to a')).$$

Case 1: K is even. Then, since $\mu_\pi(C(a \to a')) \geq qn \geq \frac{3K-4}{4K-4}n$ for all $a' \in A \setminus \{a\}$ we have that

$$W_\pi(a) \geq n\frac{3K-4}{4}. \tag{1}$$

Assume by contradiction that a is not socially acceptable. Then, there are proportions α and $1-\alpha$, with $\alpha < 1/2$ such that a fraction α of individuals places a above the line and a fraction $1-\alpha$ places a below the line. Those voters who place a above the line prefer a to at most $K-1$ alternatives. Those who place a below the line prefer a to at most $K/2-1$ alternatives. As a result, we have that

$$\begin{aligned} W_\pi(a) &\leq \alpha n(K-1) + (1-\alpha)n\big(\frac{K}{2}-1\big) \\ &= \alpha n\frac{K}{2} + n(\frac{K}{2}-1) \\ &< \frac{n}{2}\frac{K}{2} + n(\frac{K}{2}-1) \\ &= n\frac{3K-4}{4} \end{aligned}$$

where we have used the fact that $\alpha < 1/2$. This inequality contradicts inequality (1). Hence the conclusion that a is socially acceptable.

Case 2: K is odd. Then, since $\mu_\pi(C(a \to a')) \geq qn \geq \frac{3K-5}{4K-4}n$ for all $a' \in A \setminus \{a\}$ we have that

$$W_\pi(a) \geq n\frac{3K-5}{4}. \tag{2}$$

Assume by contradiction that a is not socially acceptable. Then, there are proportions α and β, with $\alpha < \beta$ and $\alpha < 1/2$, such that a fraction α of individuals places a above the line and a fraction β places a below the line. Those voters who place a above the line prefer a to at most $K-1$ alternatives. Those who place a below the line prefer a to at most $(K-3)/2$ alternatives. And those who place a on the line prefer a to exactly $(K-1)/2$ alternatives. As a result, we have that

$$\begin{aligned}
W_\pi(a) &\leq \alpha n(K-1) + \beta n(\frac{K-3}{2}) + (1-\alpha-\beta)n\frac{K-1}{2} \\
&= \alpha n(K-1) + (1-\alpha)n\frac{K-1}{2} - \beta n \\
&< \alpha n(K-1) + (1-\alpha)n\frac{K-1}{2} - \alpha n \\
&= \alpha n\frac{K-3}{2} + n\frac{K-1}{2} \\
&< \frac{n}{2}\frac{K-3}{2} + n\frac{K-1}{2} \\
&= n\frac{3K-5}{4}
\end{aligned}$$

where we have used the facts that $\alpha < \beta$ and $\alpha < 1/2$. This inequality contradicts inequality (2). Hence the conclusion that a is socially acceptable. □

The bound $\overline{q}_K$ cannot be improved. To see this, let K be even and let $q < (3K-4)/(4K-4)$. We will construct a preference profile for which alternative a is a q-Condorcet winner but is not socially acceptable. Let m be a positive integer such that $q < \big((3K-4)m + (K-2)\big)/\big((4K-4)m + 2(K-1)\big)$. Since this last expression tends to $(3K-4)/(4K-4)$ as m tends to ∞, such m can be found. Let π be the following preference profile:

		Preference	# of voters
$\succ_1$	$=$	$(a_1, a_2, \ldots, a_{\frac{K}{2}}, \underbrace{a_K, a_{\frac{K}{2}+1}, \ldots, a_{K-2}, a_{K-1}}_{\text{below the line}})$	$m+1$
$\succ_2$	$=$	$(a_2, a_3, \ldots, a_{\frac{K}{2}+1}, \underbrace{a_K, a_{\frac{K}{2}+2}, \ldots, a_{K-1}, a_1}_{\text{below the line}})$	$m+1$
$\succ_3$	$=$	$(a_3, a_4, \ldots, a_{\frac{K}{2}+2}, a_K, a_{\frac{K}{2}+3}, \ldots, a_1, a_2)$	$m+1$
$\vdots$		$\vdots$	$\vdots$
$\succ_{K-1}$	$=$	$(a_{K-1}, a_1, \ldots, a_{\frac{K}{2}-1}, a_K, a_{\frac{K}{2}}, \ldots, a_{K-2})$	$m+1$
$\succ_K$	$=$	$(a_K, \ldots)$	$m(K-1)$

Preference relation $\succ_1$ places alternative a_K in the $(K/2+1)$-th rank (right below the line), and alternatives $a_1, \ldots, a_{K-1}$ fill the remaining ranks from top to bottom, in that order. For $i = 1, \ldots, K-2$, preference relation $\succ_{1+i}$ is obtained from $\succ_i$, by maintaining a_K in the $(K/2+1)$-th rank and by "rotating" the other alternatives clockwise. Finally, there are $m(K-1)$ individuals who place a_K on top of their preference relation. The number of individuals is $n = 2(K-1)m + (K-1)$, out of whom only $m(K-1)$ place alternative a_K above the line. Therefore, a_K is not socially acceptable. On the other hand, for $k = 1, \ldots, K-1$, the number of individuals who prefer a_K to a_k is $\mu_\pi(C(a_K \to a_k)) = (K-1)m + (K/2-1)(m+1)$. As a result, the proportion of individuals that prefer a_K to any other alternative is

$$\frac{\mu_\pi(C(a_K \to a_k))}{n} = \frac{(3K-4)m + (K-2)}{(4K-4)m + 2(K-1)} > q,$$

which means that a_K is a q-Condorcet winner.

The case for odd K is similar and is left to the reader.

3 Single-peaked preferences

In this section, we restrict attention to single-peaked preferences and show that in this case, any Condorcet winner is socially acceptable.[2]

Definition 4 Let A be a set of K alternatives and let $\leq$ be a linear order on A. We say that the preference relation $\succ$ is single-peaked with respect to $\leq$ if there is an alternative $p \in A$ such that[3]

$$(a < b \leq p \text{ or } p \leq b < a) \Rightarrow b \succ a.$$

Theorem 1 Let $\leq$ be the linear order on A and assume without loss of generality that $a_1 < \cdots < a_K$. Also let π be a preference profile of single-peaked preferences with respect to $\leq$, and let $a \in A$ be a Condorcet winner for π. Then a is socially acceptable for π.

Proof : Case 1: $a \neq a_{(K+1)/2}$.

Case 1.1: $a = a_k$ for some $k \in \{1, \ldots, \lceil \frac{K-1}{2} \rceil\}$. Let $b = a_{\lceil \frac{K-1}{2} \rceil + k}$ and let $\succ$ be a preference relation in the profile. It cannot be the case that both a and b are placed above the line by $\succ$. For, in that case, since preferences are single-peaked with respect to $\leq$, a_ℓ would be above the line for all $\ell = k, \ldots, \lceil \frac{K-1}{2} \rceil + k$. But this means that more than $K/2$ alternatives would be above the line, which is impossible. On the other hand, it cannot be the case that neither a nor b are placed above the line by $\succ$. For, in that case, since preferences are single-peaked with respect to $\leq$, the number of alternatives placed above the line by $\succ$ would be at most $(\lceil \frac{K-1}{2} \rceil + k) - 1 - k) = \lceil \frac{K-3}{2} \rceil$ which is less than $\frac{K-1}{2}$, which is impossible. As a result, $\succ$ places a above the line if and only if $a \succ b$. Consequently, the number of voters who place a above the line

[2]Single-peaked preferences were introduced by Black [2].

[3]For any two alternatives, $a, b \in A$, $a < b$ means $a \leq b$ and not $b \leq a$.

equals the number of voters in the profile who prefer a to b. Since a is a Condorcet winner, this number is at least $n/2$ and therefore it is at least as large as the number of voters that place a below the line. In other words, a is socially acceptable for π.

Case 1.2: $a = a_k$ for some $k \in \{\lfloor \frac{K+1}{2} \rfloor + 1, \ldots, K\}$. Let $b = a_{k - \lceil \frac{K-1}{2} \rceil}$ and let $\succ$ be a preference relation in the profile. It cannot be the case that both a and b are placed above the line by $\succ$. For, in that case, since preferences are single-peaked with respect to $\leq$, a_ℓ would be above the line for all $\ell = k - \lceil \frac{K-1}{2} \rceil, \ldots, k$. But this means that more than $K/2$ alternatives would be above the line, which is impossible. On the other hand, it cannot be the case that neither a nor b are placed above the line by $\succ$. For, in that case, since preferences are single-peaked with respect to $\leq$, the number of alternatives placed above the line by $\succ$ would be at most $(k-1)-(k-\lceil \frac{K-1}{2} \rceil) = \lceil \frac{K-3}{2} \rceil$ which is less than $\frac{K-1}{2}$, which is impossible. As a result, $\succ$ places a above the line if and only if $a \succ b$. Consequently, the number of voters who place a above the line equals the number of voters in the profile who prefer a to b. Since a is a Condorcet winner, this number is at least $n/2$ and therefore it is at least as large as the number of voters that place a below the line. In other words, a is socially acceptable for π.

Case 2: $a = a_{(K+1)/2}$. Then, alternative a cannot be below the line for any of the voters. For suppose that $\text{rank}_\succ(a_{\frac{K+1}{2}}) > \frac{K+1}{2}$ for some preference relation $\succ$ that is single-peaked with respect to $\leq$. Then we must have that $\text{rank}_\succ(a_k) > (K+1)/2$ either for all $k \leq (K+1)/2$ or for all $k \geq (K+1)/2$. In either case we would have that more than half of the alternatives have a rank higher than $K/2$, which is impossible. As a result, a is socially acceptable for π. □

4 Single-Crossing Preferences

We now restrict attention to the class of preferences that satisfy the single-crossing property. This class has been introduced by Roberts [8] and has been shown to admit a majority voting equilibrium. See, for instance, Rothstein [9, 10], Gans and Smart [4], as well as Saporiti and Tohmé [12]. Roughly speaking, a set of preferences satisfies the single-crossing property if both the preferences and the alternatives can be ordered from "left" to "right" so that if a rightist preference prefers an alternative that is to the left of another alternative, then so do all the preferences that are to the left of this preference.

Definition 5 Let $\leq$ be a linear order on A. Also let $C \subseteq \mathcal{P}$ be a nonempty subset of preferences and $\sqsubseteq$ be a linear order on C. We say that the preference relations in C satisfy the *single-crossing property with respect to* $(\leq, \sqsubseteq)$ if for all pairs of alternatives $a, b \in A$ and for all pairs of preferences $\succ, \succ' \in C$, we have[4]

$$\left.\begin{array}{c} a < b \\ \succ \sqsubset \succ' \end{array}\right\} \Rightarrow (b \succ a \Rightarrow b \succ' a).$$

We also say that the profile $\pi = \{\succ_1, \ldots, \succ_n\}$ *satisfies the single-crossing property* if there is a linear order $\leq$ on A and a linear order $\sqsubseteq$ on the set $\pi(N)$ of preferences in the profile, such that the preferences in $\pi(N)$ satisfy the single-crossing property with respect to $(\leq, \sqsubseteq)$.

Example 2 Let the set of alternatives be $A = \{a, b, c\}$ with the linear order given by $a < b < c$. Consider the subset $C \subseteq \mathcal{P}$ that contains the following four preference relations:

$$\begin{array}{rclcrcl} \succ_1 & = & abc & \quad & \succ_2 & = & acb \\ \succ_3 & = & cab & \quad & \succ_4 & = & cba \end{array}$$

with the linear order given by $\succ_1 \sqsubset \succ_2 \sqsubset \succ_3 \sqsubset \succ_4$. It can be checked that the preferences in C satisfy the single-crossing property with respect to $(\leq, \sqsubseteq)$. Indeed, note that the preferences in C that rank alternative c over alternative a are $\succ_3$ and $\succ_4$. Similarly,

[4]For any two preferences, $\succ, \succ' \in \mathcal{P}$, $\succ \sqsubset \succ'$ b means $\succ \sqsubseteq \succ'$ and not $\succ' \sqsubseteq \succ$.

the preferences in C that rank alternative c over alternative b are $\succ_2$, $\succ_3$ and $\succ_4$. Finally, the only preference in C that ranks alternative b over alternative a is $\succ_4$. The reader can check that the preferences in C are not single-peaked with respect to any linear order on A.

The following claim states a useful property of single-crossing preferences. Namely, if two individuals agree on the ranking of two alternatives, so do all individuals who are ideologically "between" them.

Claim 1 Let $\pi = \{\succ_1, \ldots, \succ_n\}$ be a profile of preferences that are single-crossing with respect to $(\leq, \sqsubseteq)$ for some linear orders $\leq$ on A and $\sqsubseteq$ on $\pi(N)$. Also, let $i, j, k \in N$ with $\succ_i \sqsubset \succ_j \sqsubset \succ_k$. Then, for any two alternatives $a, b \in A$,

$$\text{if both } a \succ_i b \text{ and } a \succ_k b, \text{ then also } a \succ_j b.$$

Proof : Assume that $b \succ_j a$. If $a < b$, by single-crossing we must have that $b \succ_k a$. If $b < a$, by single-crossing we must have that $b \succ_i a$. □

When a set of preferences is ordered by $\sqsubseteq$, one can define its median. Formally,

Definition 6 Let $\pi = \{\succ_1, \ldots, \succ_n\}$ be a profile of preferences and let $\sqsubseteq$ be a linear order on $\pi(N)$. We say that $\succ_m \in \pi(N)$ is a median preference relation of π if

$$\mu_\pi(\{\succ \in \pi(N) : \succ \sqsubseteq \succ_m\}) \geq n/2 \quad \text{and} \quad \mu_\pi(\{\succ \in \pi(N) : \succ_m \sqsubseteq \succ\}) \geq n/2.$$

In other words, $\succ_m$ is a median preference of π if it belongs to $\pi(N)$, and at least half of the individuals have preferences that are at least as to the "right" as $\succ_m$ and at least half of the individuals have preferences that are at least as to the "left" as $\succ_m$.

It is clear that any preference profile that satisfies the single-crossing property has a median preference. The next two lemmas state that the top alternative of such a profile is both socially acceptable and a Condorcet winner.

Lemma 1 Let π be a preference profile that satisfies the single-crossing property and let $\succ_m \in \pi(N)$ be a median preference of π. Then, its top alternative is socially acceptable.

Proof : Let a be the top alternative of the median preference $\succ_m$ and let i be an individual who ranks a at least as low as any other individual. That is, $\text{rank}_{\succ_i}(a) \geq \text{rank}_{\succ_j}(a)$ for all $j \in N$. If $\text{rank}_{\succ_i}(a) \leq (K+1)/2$, a is socially acceptable since no individual places it below the line. So assume that $r = \text{rank}_{\succ_i}(a) > (K+1)/2$. There are $r-1 \geq K/2 > (K-1)/2$ alternatives $b_1, \ldots, b_{r-1}$ such that $b_k \succ_i a$ for $k = 1, \ldots, r-1$. Assume that $\succ_i \sqsubset \succ_m$. The case where $\succ_m \sqsubset \succ_i$ is similar and is left to the reader. Since $a \succ_m b_k$ for $k = 1, \ldots, r-1$ and since preferences are single-crossing, by Claim 1 we must have that $a \succ_j b_k$ for $k = 1, \ldots, r-1$, also for all $j \in N$ such that $\succ_m \sqsubseteq \succ_j$. Consequently, since such individuals constitute at least half of the voters and since $r-1 \geq K/2$, a is placed above the line by at least half of the individuals. Namely, it is socially acceptable. $\square$

Lemma 2 Let π be a preference profile that satisfies the single-crossing property and let $\succ_m \in \pi(N)$ be a median preference, and let a be its top alternative. Then a is a Condorcet winner. Furhtermore, if $\succ_m$ is the the only median preference of π, then a is the only Condorcet winner.

Proof : Let a be preference $\succ_m$'s top alternative and let b be another alternative. If $b < a$, by the single-crossing property, $a \succ_i b$ for all i such that $\succ_m \sqsubseteq \succ_i$. And if $a < b$, $a \succ_i b$ for all i such that $\succ_i \sqsubseteq \succ_m$. In either case, a is preferred to b by at least half the individuals, and therefore it is a Condorcet winner. Assume further that b is another Condorcet winner, and assume without loss of generality that $a < b$. Then, by Claim 1 and since $\succ_m$ is a median preference of π, we have that

$$1/2 \leq \mu_\pi(C(b \to a)) \leq \mu_\pi(\{\succ \in \pi(N) : \succ_m \sqsubset \succ\}) \leq 1/2.$$

This implies that the $\sqsubset$-minimal preference in $\{\succ \in \pi(N) : \succ_m \sqsubset \succ\}$ is another median preference of π. $\square$

We are now ready to state our second result.

Theorem 2 Let π be a preference profile that satisfies the single-crossing property and let $a \in A$ be a Condorcet winner with respect to π. Then a is socially acceptable for π.

Proof : Let $(\leq, \sqsubseteq)$ be the linear orders on A and $\pi(N)$, respectively, with respect to which π is single-crossing.

Let a be a Condorcet winner. If a is the top alternative for some median preference, by Lemma 1 it is socially acceptable. Therefore assume that a is not the top preference for any median preference. Since by Lemma 2 the top alternative of any median preference is a Condorcet winner, we conclude that there are multiple Condorcet winners. This also implies, by Lemma 2, that there are exactly two different median preferences in π, since there cannot be more than two median preferences in $\pi(N)$. Denote them by $\succ_{m_1}$ and $\succ_{m_2}$ and assume without loss of generality that $\succ_{m_1} \sqsubset \succ_{m_2}$. Notice that there is no preference relation $\succ_i \in \pi(N)$ such that $\succ_{m_1} \sqsubset \succ_i \sqsubset \succ_{m_2}$. Otherwise, $\succ_i$ would be a third median preference in π, which is impossible.

Let $\succ_m$ be the preference relation that is obtained from $\succ_{m_1}$ by sending a to the top. That is, $a \succ_m b$ for all $b \in A \setminus \{a\}$ and $b \succ_m b' \Leftrightarrow b \succ_{m_1} b'$ for all $b, b' \in A \setminus \{a\}$. Also let π' be the preference profile that is obtained from π by adding a single voter with preference $\succ_m$, and extend the order $\sqsubseteq$ to $\pi'(N)$ by setting $\succ_{m_1} \sqsubset \succ_m \sqsubset \succ_{m_2}$. It can be checked that profile π' satisfies the single-crossing property with respect to $\leq$ and the extended order $\sqsubseteq$. To see this, it is enough to check comparisons only involving a since, restricted to $A \setminus \{a\}$, profiles π and π' contain the same preferences. Let $b \in A \setminus \{a\}$ and let $\succ_i \in \pi(N)$. We know that $a \succ_m b$. Assume first that $b < a$

and $\succ_m \sqsubset \succ_i$. We need to show that $a \succ_i b$. If $a \succ_{m_1} b$, then by the single-crossing property of π, $a \succ_i b$. If, on the other hand, $b \succ_{m_1} a$, we must have that $a \succ_{m_2} b$, because otherwise, if $b \succ_{m_2} a$, by the single-crossing property of π we would have that $b \succ_j a$ for all $\succ_j \in \pi(N)$ such that $\succ_j \sqsubset \succ_{m_2}$. Since $\succ_{m_2}$ is the "right" median preference, this means that more than half of the individuals would prefer b to a, which contradicts the fact that a is a Condorcet winner. Summarizing, we have that $b < a$ and $a \succ_{m_2} b$. Then, by the single-crossing property of π, $a \succ_j b$ for all $\succ_j \in \pi(N)$ such that $\succ_{m_2} \sqsubseteq \succ_j$. In particular $a \succ_i b$, which is what we wanted to show. Similarly, assume now that $a < b$. Since $a \succ_m b$, we need to show that $a \succ_i b$ for all $\succ_i$ such that $\succ_i \sqsubset \succ_m$. If $a \succ_{m_2} b$, then by the single-crossing property of π, $a \succ_i b$ for all $\succ_i$ such that $\succ_i \sqsubset \succ_m$. If, on the other hand, $b \succ_{m_2} a$, we must have that $a \succ_{m_1} b$, because otherwise, if $b \succ_{m_1} a$, by the single-crossing property of π we would have that $b \succ_j a$ for all $\succ_j \in \pi(N)$ such that $\succ_{m_1} \sqsubset \succ_j$. Since $\succ_{m_1}$ is the "left" median preference, this means that more than half of the individuals would prefer b to a, which contradicts the fact that a is a Condorcet winner. Summarizing, we have that $a < b$ and $a \succ_{m_1} b$. Then, by the single-crossing property of $\pi(N)$, $a \succ_i b$ for all $\succ_i \in \pi(N)$ such that $\succ_i \sqsubseteq \succ_{m_1}$. We conclude that π' satisfies the single-crossing property.

We can now prove that a is socially acceptable for π. The proof is similar to that of Lemma 1. By construction, $\succ_m$ is a median preference of π' and a is its top alternative. Let i be an individual who ranks a at least as low as any other individual. That is, $\text{rank}_{\succ_i}(a) \geq \text{rank}_{\succ_j}(a)$ for all $j \in N$. If $\text{rank}_{\succ_i}(a) \leq (K+1)/2$, a is socially acceptable since it is not placed below the line by any individual. Therefore, assume that $r = \text{rank}_{\succ_i}(a) > (K+1)/2$. Then, there are $r - 1 \geq K/2 > (K-1)/2$ alternatives $b_1, \ldots, b_{r-1}$ such that $b_k \succ_i a$ for $k = 1, \ldots, r-1$. Assume that $\succ_i \sqsubset \succ_m$. The case where $\succ_m \sqsubset \succ_i$ is similar and is left to the reader. Since preferences are single-crossing for π', and $a \succ_m b_k$ for $k = 1, \ldots, r-1$, by Claim 1 we must have that $a \succ_j b_k$ for $k = 1, \ldots, r-1$, for all $j \in N$ such that $\succ_m \sqsubseteq \succ_j$. One such individual is m_2. Therefore, $a \succ_j b_k$ for $k = 1, \ldots, r-1$, for all $j \in N$ such that $\succ_{m_2} \sqsubseteq \succ_j$.

Since $r-1 \geq K/2$, all these individuals place a above the line. Since such individuals constitute at least half of the voters in π, a is socially acceptable. □

5 Other preference domains

In this section we look at two classes of preferences that admit Condorcet winners; single-dipped preferences and group-separable preferences.

For any preference relation $\succ$, its inverse $\succ^{-1}$ is the preference relation that for all $a, b \in A$, $a \succ^{-1} b$ if and only if $b \succ a$. We say that a $\pi = (\succ_1, \ldots, \succ_n)$ is a profile of single-dipped preferences if the profile of its inverses $(\succ_1^{-1}, \ldots, \succ_n^{-1})$ is single-peaked with respect to some linear order $<$ of the alternatives. Single-dipped preference profiles admit a Condorcet winner. This follows from the fact that the top alternative of each voter is either the first or the last alternative in the linear order $<$, and therefore one of these two alternatives must be preferred to all the other alternatives by at least half of the individuals. As a result, a Condorcet winner of a single-dipped preference profile must be socially acceptable since, being the top alternative of at least half the individuals, it is placed above the line by them.

Another family of preferences that admit a Condorcet winner is the family of group-separable preferences introduced by Inada [6]. A set of preferences $\mathcal{P}'$ is *group-separable* if every subset $A' \subseteq A$ with at least three alternatives can be partitioned into two non-empty subsets A_1 and A_2 such that for each preference relation $\succ \in \mathcal{P}'$, either $a_1 \succ a_2$ for all $a_1 \in A_1$ and all $a_2 \in A_2$, or $a_2 \succ a_1$ for all $a_1 \in A_1$ and all $a_2 \in A_2$. The partition $\{A_1, A_2\}$ is said to be a *group-separable partition* of A'.

It turns out that a Condorcet winner for a group-separable preference profile may well not be socially acceptable. To see this, let $A = \{a, b, c, d, e\}$ be a set of five alternatives and consider the set of preferences $\mathcal{P}' = \{\succ_1, \ldots, \succ_5\}$ where $\succ_1 =$ *abced*, $\succ_2 =$ *cbaed*, $\succ_3 =$ *debac*, $\succ_4 =$ *decab*, and $\succ_5 =$ *decba*. This set of preferences are group-separable. Indeed, $\{\{a,b,c\},\{d,e\}\}$ is a group-separable partition of A, and $\{\{a,b\},\{c\}\}$ is a group-separable partition of $\{a,b,c\}$. Consider a preference profile containing five copies of $\succ_1$, and two copies of each of the other preference relations in $\mathcal{P}'$. It can be checked that whereas a is a Condorcet winner for this profile it is not socially acceptable. Indeed, six out of the thirteen voters place a below the line and only five voters place it above the line.

6 Maximal families of preferences

We have seen that any profile of preferences that are single-peaked preferences with respect to some linear order is regular. However, the family of single-peaked preferences with respect to some linear order is not maximal in the sense that one can find another set of preferences strictly containing this family such that any profile of preferences from it is regular. The same can be said about the family of preferences that are single-crossing. In this section, we are interested in identifying large families of preferences such that any profile built from them is regular, and such that they are maximal with respect to this property.

As mentioned earlier, when the number of alternatives K is three, all preference profiles are regular. Assume then that K is bigger than 3. Let $\mu : A \to A$ be a one-to-one function such that $\mu(a) \neq a$ for all $a \in A$, and define $\mathcal{P}^\mu$ to be the set of preferences $\succ$ such that for all $a \in A$, a is placed above the line by $\succ$ whenever $a \succ \mu(a)$. The next proposition states that any Condorcet winner of a profile composed of preferences taken from $\mathcal{P}^\mu$ is socially acceptable. Furthermore, $\mathcal{P}^\mu$ is maximal with respect to this property.

Theorem 3 Any profile of preferences taken from $\mathcal{P}^\mu$ is regular. Furthermore, for any subset $\widehat{\mathcal{P}}$ of preferences that strictly contains $\mathcal{P}^\mu$, there is a profile of preferences taken from it with a Condorcet winner that is not socially acceptable.

Proof : Consider a profile π of preferences from $\mathcal{P}^\mu$. It can be seen that if alternative a is not socially acceptable, it is not a Condorcet winner either. The reason is that if a is not socially acceptable, more than half of the voters place a on or below the line. As a consequence, since their preferences belong to $\mathcal{P}^\mu$, all of these voters prefer $\mu(a)$ to a. We now show that $\mathcal{P}^\mu$ is maximal with respect to this property. Namely, for any subset $\widehat{\mathcal{P}}$ of preferences that strictly contains $\mathcal{P}^\mu$, we can build a preference profile from it containing a Condorcet winner which is not socially acceptable. To see this, let $\succ'$ be a preference relation that, for some alternative a, places a below or on the line, but $a \succ' \mu(a)$. Let $A' = \{b_1, \ldots, b_\kappa\}$ be the set of alternatives b such that $b \succ' a$. Since $\succ'$ does not place a above the line, $\kappa \geq \lfloor K/2 \rfloor$. Construct a profile $\pi = (\succ_1, \ldots, \succ_{2\kappa+2})$ of preferences taken from $\mathcal{P}^\mu \cup \{\succ'\}$ as follows. Preference $\succ_i$, for $i = 1, \ldots, \kappa$, places a on top. For each alternative b_i in A' preference $\succ_{\kappa+i}$ belongs to $\mathcal{P}^\mu$, places a below the line, and is such that $a \succ_{\kappa+i} b_i$. (We will soon show that such a preference exists.) Preference $\succ_{2\kappa+1}$ is a replica of $\succ_{2\kappa}$, and lastly, preference $\succ_{2\kappa+2}$ is $\succ'$. Alternative a is not socially acceptable for this profile; indeed, preferences $\succ_{\kappa+1}, \ldots, \succ_{2\kappa+1}$ place it below the line while only preferences $\succ_1, \ldots, \succ_\kappa$ place it above the line. On the other hand, a is a Condorcet winner. To see this note that for each $b_i \in A'$, there are at least $\kappa+1$ voters that prefer a to b_i (voters $1, \ldots, \kappa$ and voter $\kappa + i$). And for each of the remaining alternatives, $b \in A \backslash A'$, $b \neq a$, there are also at least $\kappa + 1$ voters that prefer a to them (voters $1, \ldots, \kappa$ and voter $2\kappa + 2$).

It remains to show that for each alternative b_i in A' there is a preference relation $\succ_{\kappa+i} \in \mathcal{P}^\mu$ which places a below the line and is such that $a \succ_{\kappa+i} b_i$. We now build such a preference relation. First note that since μ is one-to-one, we can partition A into M nonempty sets $A_m = \left\{a_1^m, \ldots, a_{k_m}^m\right\}$, $m = 1, \ldots, M$ such that $\mu(a_j^m) = a^m_{j+1 \text{mod}(k_m)}$. Since for no alternative a' do we have that $\mu(a') = a'$, each subset A_m has at least two alternatives and therefore, $M \leq K/2$. Without loss of generality assume that $a \in A_1$. There are two cases to consider.

Case 1: $b_i \notin A_1$. Assume without loss of generality that $b_i \in A_M$ and that $b_i = a_1^M$.

Also assume that $a = a_1^1$. In this case we build $\succ_{\kappa+i}$ as follows. Place alternatives $a_{k_1}^1, \ldots, a_{k_M}^M$ in the first M ranks of $\succ_{\kappa+i}$. Then, place alternatives $a_1^M = b_i$ and $a_1^1 = a$ in the last two ranks of $\succ_{\kappa+i}$, so that $a \succ_{\kappa+i} b_i$. Finally, fill the ranks of $\succ_{\kappa+i}$ from the bottom up with the remaining alternatives so that for $m = 1, \ldots, M$, $a_{j+1}^m \succ_{\kappa+i} a_j^m$ for $j = 1, \ldots, k_m$. The order just constructed belongs to $\mathcal{P}^\mu$ (because the alternatives are placed so that $a_{j+1}^m \succ_{\kappa+i} a_j^m$), places a below the line, and is such that $a \succ_{\kappa+i} b_i$.

Case 2: $b_i \in A_1$. Assume without loss of generality that $b_i = a_1^1$. Since $b_i \in A'$, $b_i \neq \mu(a), a$. Therefore $a = a_s^1$ for some $1 < s < k_1$.

Case 2.1: $s < (K+1)/2$. In this case we build $\succ_{\kappa+i}$ as follows. Place alternatives $a_{k_1}^1, a_{k_2}^2, \ldots, a_{k_M}^M$ in the first M ranks of $\succ_{\kappa+i}$. Then start filling the ranks of $\succ_{\kappa+i}$ from the bottom up with the alternatives $b_i = a_1^1, a_2^1, \ldots, a = a_s^1, \ldots, a_{k_1-1}^1$. Finally, fill the ranks of $\succ_{\kappa+i}$ from the bottom up with the remaining alternatives so that for $m = 2, \ldots, M$, $a_{j+1}^m \succ_{\kappa+i} a_j^m$ for $j = 1, \ldots, k_m$. The order just constructed places a below the line (because $s < (K+1)/2$), belongs to $\mathcal{P}^\mu$ (because the alternatives are placed so that $a_{j+1}^m \succ_{\kappa+i} a_j^m$), and is such that $a \succ_{\kappa+i} b_i$ (because b_i is at the bottom of the preference relation).

Case 2.2: $s \geq (K+1)/2$. In this case we build $\succ_{\kappa+i}$ as follows. Place alternative $a_{s+1} = \mu(a)$ in rank $\lceil K/2 \rceil$ (on or right above the line). Then, place a in rank $\lceil K/2+1 \rceil$ (right below the line). Then start filling the ranks of $\succ_{\kappa+i}$ from the bottom up with the alternatives $b_i = a_1^1, a_2^1, \ldots, a_{\lfloor \frac{K}{2}-1 \rfloor}^1$, in that order. Finally, fill the remaining ranks of $\succ_{\kappa+i}$ (all of which are above the line) with all the remaining alternatives. The order just constructed belongs to $\mathcal{P}^\mu$ (because the alternatives placed below or on the line are, from the bottom up, $a_1^1, a_2^1, \ldots, a_{\lfloor \frac{K}{2}-1 \rfloor}, a_s^1, a_{s+1}^1$, in that order), places a right below the line and is such that $a \succ_{\kappa+i} b_i$ (because b_i is at the bottom of the preference relation). □

6.1 Discussion

a) The size of the set $\mathcal{P}^\mu$ depends on μ but can be large. It is typically much larger than the cardinality of the set of preferences that are single-peaked (2^{K-1}), and that of the set of preferences with the single-crossing property ($K(K-1)/2+1$). For instance, when $A=\{a,b,c,d\}$ and $\mu : A \to A$ is defined by $\mu(a)=b$, $\mu(b)=a$, $\mu(c)=d$, $\mu(d)=c$, the class $\mathcal{P}^\mu$ contains sixteen out of the twenty-four preference relations. On the other hand, due to its size, $\mathcal{P}^\mu$ is not generally a Condorcet domain, namely profiles of preferences taken from it may have no Condorcet winner.

b) When the number of alternatives K is even, the set of single peaked preferences with respect to $a_1 < a_2 < \cdots < a_K$ is a subset of the family $\mathcal{P}^\mu$, where μ is defined by $\mu(a_k) = a_{\frac{K}{2}+k}$ and $\mu(a_{\frac{K}{2}+k}) = a_k$, for $k=1,\ldots,K/2$. However, when K is odd, the family of single-peaked preferences with respect to any linear order $<$ of the alternatives is not contained in $\mathcal{P}^\mu$ for any μ. To see this, consider $A=\{a,b,c\}$. It can be checked that preferences abc and cba are single-peaked with respect to the linear order $a<b<c$. However, no matter how $\mu(b)$ is defined, a preference profile containing these two preferences will not satisfy that b is above the line whenever b is preferred to $\mu(b)$. Similarly, the preference profile $\{abc, cab\}$ is single-crossing with respect to $a<b<c$ and $abc \sqsubseteq cab$ but no matter how $\mu(b)$ is defined, it does not satisfy that b is above the line whenever b is preferred to $\mu(b)$. Therefore we conclude that the family of preferences that satisfy the single-crossing property is not contained in $\mathcal{P}^\mu$ for any μ. Similar examples for the case where K is odd and bigger than 3 can be readily built, and are left to the reader.

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