Beauty Contests: Beyond Gaussian Uncertainty*

Oscar Volij

Ben Gurion University of the Negev.

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Abstract

We analyze a beauty contest model within a framework of bounded action and signal spaces, and quadratic payoffs. We show conditions that guarantee that the symmetric equilibrium strategy is affine and monotone in the signals. We illustrate the results with several information structures. We also compute the (not necessarily affine) equilibrium strategy in a general mixture signal structure. *Journal of Economic Literature* Classification Numbers: C72, D83.

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*Email address: ovolij@bgu.ac.il.

1 Introduction

Beauty contest models are Bayesian games in which agents choose actions based on private and public information about an underlying fundamental, and where payoffs reflect a tradeoff between matching the fundamental and coordinating with others. A central insight of that literature, unveiled by Morris and Shin (2002), is that improved information about the fundamental may worsen welfare by inducing excessive sensitivity to private signals and thereby increasing dispersion in actions. Subsequent work by Angeletos and Pavan (2007) analyzes the welfare properties of the equilibrium. Myatt and Wallace (2012) adds costly information acquisition to the beauty contest model and asks which signals tend to receive more attention. These models frequently assume Gaussian distributions for both the underlying fundamental state and the players' signals. While the Gaussian framework is mathematically convenient because it yields linear posterior expectations, it requires unbounded state spaces and assumes that the variance of the noise is independent of the true state. This paper explores how the mechanics of the beauty contest adapt when these standard distributional assumptions are relaxed. The paper examines a general beauty contest game where players receive both private and public signals, and react to them by choosing actions from a bounded set. We represent the symmetric Bayesian equilibrium as the solution to a functional equation and establish the existence and uniqueness of the equilibrium strategy. We uncover properties of the equilibrium by identifying classes of functions that are invariant under the best-response operator. This approach yields several results. Under affiliation, equilibrium strategies are monotone in signals. When conditional expectations are affine, equilibrium strategies are affine. Most importantly, in mixture signal structures, equilibrium strategies are linear in posterior beliefs. In the finite-state case, this leads to an explicit representation in terms of a system of linear equations. We illustrate these results using specific signal structures. In a model with a uniform prior and binomial signals, we show that equilibrium strategies are affine and that increasing the size of the sample from which the signals are obtained, increases both the accuracy of the players' actions and their coordination, with the resulting increase in equilibrium payoffs. In a parametric model, increased informativeness of signals reduces the mean squared error relative to the fundamental but increases dispersion across agents, generating a tradeoff between accuracy and coordination. Despite this tradeoff, overall equilibrium payoffs increase with informativeness. Finally, a simple mixture signal model with piecewise linear posteriors, highlights a feature absent from Gaussian models: signal insensitivity in the tails. In this setting, extreme signal realizations lead to saturated posterior beliefs, causing players to ignore marginal variations in extreme signals and keeping their actions strictly bounded.

2 A beauty contest

We begin by describing the model and establishing existence and uniqueness of a symmetric equilibrium strategy. The key step is to formulate the best-response operator as a contraction on a suitable space of functions. Let $\mathcal{S}$, $\mathcal{X}$, and $\mathcal{Y}$ be three compact subsets of the real line. The first is the set of possible states of nature, the second is the set of possible private signals, and the third is the set of possible realizations of a public signal. There are n players, each of whom observes a private signal $x \in \mathcal{X}$, and all players observe a common public signal $y \in \mathcal{Y}$. Thus, the set of states of the world is

$$\Omega = \mathcal{S} \times \mathcal{X}^n \times \mathcal{Y}.$$

Let μ be a probability measure on the Borel sets of Ω. For $\omega = (\theta, x_1, \ldots, x_n, y) \in \Omega$, define the random variables

$$\Theta(\omega) = \theta \qquad X_i(\omega) = x_i, \qquad Y(\omega) = y, \qquad i = 1, \ldots, n.$$

We assume that Θ is nondegenerate and that the private signals $X_1, \ldots, X_n$ are exchangeable conditional on Y. For each $i \in N$, $x \in \mathcal{X}$, and $y \in \mathcal{Y}$, let $\mu(\cdot \mid X_i = x, Y = y)$ denote the conditional distribution of (Θ, X_{-i}) given $(X_i = x, Y = y)$.

We assume that this conditional distribution is weakly continuous in (x, y), so that for every continuous function $g : \Theta \times X_{-i} \to \mathbb{R}$, the function $\mathrm{E}[g(\Theta, X_{-i}) \mid X_i = x, Y = y]$ is continuous in x and y.

After the players observe the realization of their respective signals, they simultaneously choose an action from their sets of actions $A_i = \mathbb{R}$, $i = 1, \ldots, n$. Given an action profile $a = (a_1, \ldots, a_n)$ and a realization $(\theta, x_1, \ldots, x_n, y) \in \Omega$, player i's (negative) payoff is a weighted average of the distances of the action from the fundamental and from the average action:

$$u_i(a, (\theta, x_1, \ldots, x_n, y)) = -r(a_i - \theta)^2 - (1 - r)\left(a_i - \frac{\sum_{j \in N} a_j}{n}\right)^2 \tag{1}$$

where $r \in (0, 1)$ is the weight put on the squared distance of his action from the state of nature. An alternative payoff specification (adopted by Morris and Shin (2002)) would be

$$v_i(a, (\theta, x_1, \ldots, x_n)) = -r(a_i - \theta)^2 - (1 - r)\left(L_i - \bar{L}\right) \tag{2}$$

where

$$L_i = \sum_{j=1}^{n} \frac{(a_i - a_j)^2}{n} \qquad \bar{L} = \sum_{j=1}^{n} \frac{L_i}{n}$$

As we shall see, the analysis under both specifications is very similar.

The above description defines a Bayesian game. A strategy for player i consists of a function $\beta_i : \mathcal{X} \times \mathcal{Y} \to A_i$. A Bayesian equilibrium of this game consists of a strategy profile $(\beta_i)_{i=1}^n$ such that for all $i = 1, \ldots, n$,

$$\beta_i(x, y) = \max_{a_i \in A_i} \mathrm{E}[u_i(\left(a_i, \beta_{-i}(X_{-i}, Y)\right), (\Theta, X_1, \ldots, X_n, Y)) | X_i = x, Y = y].$$

We look for a symmetric equilibrium in which all players behave according to the same strategy $\beta : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$. Assume that all players other than i behave according to β, and consider a player who observes signals (x, y). If he chooses a_i, he gets

$$\mathrm{E}[-r\,(a_i - \Theta)^2 - (1 - r)\left(a_i - \frac{a_i + \sum_{j \neq i} \beta(X_j, Y)}{n}\right)^2 \mid X_i = x, Y = y].$$

Since the objective function is strictly concave in a_i, the first-order condition is necessary and sufficient. This implies that the optimal strategy must satisfy

$$\beta(x,y) = \frac{(1-r)(n-1)^2 \mathrm{E}[\frac{\sum_{j\neq i}\beta(X_j,Y)}{n-1}|X_i = x, Y = y] + rn^2 \mathrm{E}[\Theta|X_i = x, Y = y]}{(1-r)(n-1)^2 + rn^2}.$$

Letting $b = \frac{rn^2}{(1-r)(n-1)^2+rn^2}$, we have that for $(x,y) \in \mathcal{X} \times \mathcal{Y}$,

$$\beta(x,y) = (1-b)\mathrm{E}[\frac{\sum_{j\neq i}\beta(X_j,Y)}{n-1}|X_i = x, Y = y] + b\mathrm{E}[\Theta|X_i = x, Y = y]. \quad (3)$$

If we used the alternative payoff specification (2), the resulting functional equation would still be given by (3), except that in this case the coefficient would be $b = \frac{rn^2}{(1-r)(n-1)(n-2)+rn^2}$.

Let the expected value of the fundamental be defined by

$$m(x,y) = \mathrm{E}[\Theta|X_i = x, Y = y] \quad (4)$$

By our previous assumptions on μ, this function is continuous and, by exchangeability, it does not depend on the conditioning signal i.

Let $\mathcal{B}$ be the set of continuous real functions defined on $\mathcal{X} \times \mathcal{Y}$, endowed with the sup norm. Since $\mathcal{X} \times \mathcal{Y}$ is compact, all the functions in $\mathcal{B}$ are bounded. Define the operator $K : \mathcal{B} \to \mathcal{B}$ by

$$(K\phi)(x,y) = \mathrm{E}\Big[\sum_{j\neq i}\frac{\phi(X_j,Y)}{n-1}\;\Big|\; X_i = x, Y = y\Big].$$

By conditional exchangeability, this operator does not depend on i, and can be written as

$$(K\phi)(x,y) = \mathrm{E}[\phi(X_j,Y) \mid X_i = x, Y = y], \quad j \neq i. \quad (5)$$

This expression does not depend on the choice of $j \neq i$ by exchangeability. As a result, equation (3) becomes

$$\beta(x,y) = (1-b)(K\beta)(x,y) + b\,m(x,y) \qquad (x,y) \in \mathcal{X} \times \mathcal{Y}. \quad (6)$$

Under the continuity assumption on conditional distributions, the operator K maps $\mathcal{B}$ into itself. Also, define inductively

$$K^{t+1}\phi(x,y) = KK^t\phi(x,y) \qquad t = 1, 2, \ldots$$

The following theorem characterizes the unique symmetric equilibrium of the game.

Theorem 1. There is a unique strategy $\beta \in \mathcal{B}$ that satisfies the functional equation (6). Furthermore, this strategy is

$$\beta(x, y) = b\, m(x, y) + b \sum_{t=1}^{\infty} (1-b)^t (K^t m)(x, y).$$

Proof. For any continuous function $\phi : X \times \mathcal{Y} \to \mathbb{R}$, define

$$(T\phi)(x, y) := (1-b)(K\phi)(x, y) + b\, m(x, y).$$

Since m is continuous, T maps $\mathcal{B}$ into itself. Note that

a) if $\phi(x, y) \leq \psi(x, y)$ for all (x, y), then $(T\phi)(x, y) \leq (T\psi)(x, y)$ for all (x, y), and

b) $T(\phi + d)(x, y) = (T\phi)(x, y) + (1-b)d$ for all (x, y) and all constants $d \geq 0$.

Therefore, since $0 < b < 1$, T satisfies Blackwell's sufficient conditions for a contraction, and T is a contraction with modulus $(1-b)$. Since $(\mathcal{B}, \|\cdot\|)$ is a complete metric space, by the contraction mapping theorem, T has a unique fixed point $\beta \in \mathcal{B}$. (See for instance, Stokey, Lucas, and Prescott (1989), Theorem 3.2 and 3.3). This fixed point satisfies

$$(I - (1-b)K)\beta(x, y) = b\, m(x, y).$$

Notice that K is a linear operator. Also, since $\|K\phi\| \leq \|\phi\|$ for all $\phi \in \mathcal{B}$, we have that $\frac{\|(1-b)K\phi\|}{\|\phi\|} = \frac{(1-b)\|K\phi\|}{\|\phi\|} < 1$ for all $\phi \in \mathcal{B}$ we have that, (see Eidelman, Milman, and Tsolomitis (2004), Section 4.7) $(I - (1-b)K)$ is invertible and

$$(I - (1-b)K)^{-1} = \sum_{t=0}^{\infty} ((1-b)K)^t.$$

Thus,

$$\beta(x, y) = b \sum_{t=0}^{\infty} ((1-b)K)^t m(x, y).$$

□

A useful and intuitive property of the equilibrium strategy is that the expected action equals the expected value of the fundamental. This is stated in the next proposition.

Proposition 1. The equilibrium strategy is an unbiased estimator of the mean common value: $\mathrm{E}[\beta(X_i, Y)] = \mathrm{E}[\Theta]$.

Proof. Since $\beta(x, y) = (1 - b)(K\beta)(x, y) + b\, m(x, y)$, we have

$$\mathrm{E}[\beta(X_i, Y)] = (1 - b)\mathrm{E}[(K\beta)(X_i, Y)] + b\, \mathrm{E}[m(X_i, Y)]. \tag{7}$$

By definition of K and the law of iterated expectations,

$$\mathrm{E}[(K\beta)(X_i, Y)] = \mathrm{E}\big[\mathrm{E}[\beta(X_j, Y) \mid X_i, Y]\big] = \mathrm{E}[\beta(X_j, Y)] \quad j \neq i. \tag{8}$$

Similarly,

$$\mathrm{E}[m(X_i, Y)] = \mathrm{E}[\mathrm{E}[\Theta \mid X_i, Y]] = \mathrm{E}[\Theta]. \tag{9}$$

Substituting (8–9) into (7),

$$\mathrm{E}[\beta(X_i, Y)] = (1 - b)\mathrm{E}[\beta(X_j, Y)] + b\, \mathrm{E}[\Theta].$$

By exchangeability, $\mathrm{E}[\beta(X_j, Y)] = \mathrm{E}[\beta(X_i, Y)]$, and since $b > 0$, we obtain $\mathrm{E}[\beta(X_i, Y)] = \mathrm{E}[\Theta]$. □

For later reference, we now introduce a well-known and powerful observation that will be used throughout the paper. If a class of functions is invariant under the best-response operator and m belongs to this class, then the equilibrium strategy must belong to that class.

Observation 1. Let $\mathcal{B}'$ be a subset of continuous functions on $\mathcal{X} \times \mathcal{Y}$ such that: (i) $\mathcal{B}'$ is closed, (ii) $m(x, y) \in \mathcal{B}'$, (iii) for each $\phi \in \mathcal{B}'$, $K\phi \in \mathcal{B}'$, and (iv) closed under addition. Then the equilibrium strategy identified in Theorem 1 belongs to $\mathcal{B}'$.

Proof. Since for all $\phi \in \mathcal{B}'$, $m(x,y) \in \mathcal{B}'$ and $K\phi \subseteq \mathcal{B}'$, the transformation $(T\phi)(x,y) = (1-b)(K\phi)(x,y) + b\,m(x,y)$ maps $\mathcal{B}'$ into itself. Since $\mathcal{B}'$ is a closed subset of the complete metric space $\mathcal{B}$, it is complete. As a result, T has a unique fixed point, and it must coincide with the one identified in Theorem 1. □

This observation will be used repeatedly to characterize the structure of equilibrium strategies by identifying invariant classes of functions. We first apply this approach to monotonicity. Except for exchangeability, we have not imposed any restriction on the signals. The next proposition states that if the state of nature and the signals are affiliated, then the equilibrium strategy is non-decreasing. Affiliation is a strong form of positive correlation. The reader is referred to Milgrom and Weber (1982) or to Krishna (2002) for a formal definition.

Theorem 2. Assume that the signals $\Theta, X_1, \ldots, X_n, Y$ are affiliated. Then the unique strategy β found in Theorem 1 is nondecreasing in (x,y). If additionally Θ and (X_1, Y) are strictly affiliated, then β is strictly increasing.

Proof. Let $\mathcal{B}'$ be the set of continuous functions that are nondecreasing in (x,y). It is a closed subset of $\mathcal{B}$ and hence complete. It is also closed under addition. Let $\phi \in \mathcal{B}'$. Since $(\Theta, X_1, \ldots, X_n, Y)$ are affiliated, the functions $m(x,y)$ and $(K\phi)(x,y)$, defined in (4) and (5) are nondecreasing in (x,y) (see Milgrom and Weber (1982), Theorem 5). Therefore, by Observation 1, β is nondecreasing in (x,y).

If (Θ, X_1, Y) are strictly affiliated, then $m(x,y)$ is strictly increasing. Hence the transformation $(T\phi)(x,y) = (1-b)(K\phi)(x,y) + b\,m(x,y)$ whose fixed point is β, maps $\mathcal{B}'$ into strictly increasing functions, and therefore, since $\beta = T\beta$, β is strictly increasing. □

3 Affine functions

In this section we consider conditions under which equilibrium strategies are affine. When the conditional expectation of the fundamental and of other agents' signals are affine functions of the observed signals, the best-response operator preserves affine functions, leading to affine equilibrium strategies. This is stated in the following proposition.[1]

Proposition 2. If $m(x, y) = \mathrm{E}[\Theta \mid X_i = x, Y = y]$ and $\mathrm{E}[X_j \mid X_i = x, Y = y]$ are affine functions of (x, y), then the unique symmetric equilibrium strategy is also affine. Namely, for some α, γ and δ, $\beta(x, y) = \alpha + \gamma x + \delta y$.

Proof. Let $\mathcal{B}'$ denote the set of affine functions on $\mathcal{X} \times \mathcal{Y}$. This set is a closed subset of $\mathcal{B}$. It is also closed under addtion. Take $\phi(x, y) = \alpha + \gamma x + \delta y \in \mathcal{B}'$. Then

$$(K\phi)(x, y) = \mathrm{E}[\alpha + \gamma X_j + \delta Y \mid X_i = x, Y = y] = \alpha + \gamma \mathrm{E}[X_j \mid X_i = x, Y = y] + \delta y.$$

By assumption, $\mathrm{E}[X_j \mid X_i = x, Y = y]$ is affine in (x, y), so $(K\phi)(x, y)$ is affine. Since $m(x, y)$ is also affine by assumption, it follows Observation 1, the unique fixed point belongs to $\mathcal{B}'$, and hence β is affine. $\square$

Proposition 2 is useful since it allows us to find β by matching coefficients when we know that the equilibrium strategy is affine.

Corollary 1. Suppose that $\mathrm{E}[X_j \mid X_i = x, Y = y] = \delta_E + \eta_E x + \zeta_E y$ and $m(x, y) = \delta_m + \eta_m x + \zeta_m y$. Then the equilibrium strategy is affine, and takes the form $\beta(x, y) = \alpha + \gamma x + \delta y$, where

$$\gamma = \frac{b\,\eta_m}{1 - (1 - b)\eta_E}, \qquad \delta = \zeta_m + \frac{(1 - b)\eta_m \zeta_E}{1 - (1 - b)\eta_E}, \qquad \alpha = \delta_m + \frac{(1 - b)\eta_m \delta_E}{1 - (1 - b)\eta_E}.$$

Proof. By Proposition 2, the equilibrium strategy takes the form $\beta(x, y) = \alpha + \gamma x + \delta y$. Therefore, by the definition of K (see equation (5)), $(K\beta)(x, y) = \alpha + \gamma\,\mathrm{E}[X_j \mid X_i = x, Y = y] + \delta y$, and we obtain that $(K\beta)(x, y) = \alpha + \gamma(\delta_E + \eta_E x + \zeta_E y) + \delta y$. Substituting into the functional equation (6) gives

$$\alpha + \gamma x + \delta y = (1 - b)\big(\alpha + \gamma(\delta_E + \eta_E x + \zeta_E y) + \delta y\big) + b(\delta_m + \eta_m x + \zeta_m y).$$

[1] Diaconis and Ylvisaker (1979) state conditions for the information structure to yield linear posteriors.

Matching coefficients of 1, x, and y on both sides yields the system:

$$\alpha = (1-b)(\alpha + \gamma\delta_E) + b\delta_m,$$

$$\gamma = (1-b)\gamma\eta_E + b\eta_m,$$

$$\delta = (1-b)(\gamma\zeta_E + \delta) + b\zeta_m.$$

The solution to this system of equations yields the desired result. □

Corollary 2. If the equilibrium strategy is affine, then $\beta(x, y) = \mathrm{E}[\Theta] + \gamma(x - \mathrm{E}[X_i]) + \delta(y - \mathrm{E}[Y])$.

Proof. Assume $\beta(x, y) = \alpha + \gamma x + \delta y$. Then

$$\beta(x, y) = \alpha + \gamma\mathrm{E}[X_i] + \delta\mathrm{E}[Y] + \gamma(x - \mathrm{E}[X_i]) + \delta(y - \mathrm{E}[Y]).$$

Since by Proposition 1, $\mathrm{E}[\beta(X_i, Y)] = \mathrm{E}[\Theta]$, we have $\alpha + \gamma\mathrm{E}[X_i] + \delta\mathrm{E}[Y] = \mathrm{E}[\Theta]$, which yields the desired result. □

3.1 Expected payoffs

In this section we apply the previous results to study how the informativeness of the signal structure affects payoffs when both private and public signals are present and the equilibrium strategy is affine. We will see that when the equilibrium strategy is affine, expected payoffs depend only on second moments of the underlying variables.

If player payoffs are given by (1), the equilibrium expected payoff is

$$U = r\mathrm{E}[(\beta(X_1, Y) - \Theta)^2] + (1-r)\mathrm{E}\left[\left(\beta(X_1, Y) - \frac{1}{n}\sum_{j=1}^{n}\beta(X_j, Y)\right)^2\right].$$

If payoffs are given by the alternative specification (2) instead, the equilibrium expected payoff is

$$V = r\mathrm{E}[(\beta(X_1, Y) - \Theta)^2].$$

We first characterize the mean squared error in terms of the moments of the primitive random variables.

Claim 1. If $\beta(x,y) = \alpha + \gamma x + \delta y$, then

$$\mathrm{E}[(\beta(X_1, Y) - \Theta)^2] = \mathrm{Var}(\Theta) + \mathrm{Var}(\gamma X_1 + \delta Y) - 2\,\mathrm{Cov}(\gamma X_1 + \delta Y, \Theta).$$

Proof. By Proposition 1, $\mathrm{E}[\beta(X_1, Y)] = \mathrm{E}[\Theta]$. Therefore,

$$\begin{aligned}\mathrm{E}[(\beta(X_1, Y) - \Theta)^2] &= \mathrm{Var}(\beta(X_1, Y) - \Theta) \\ &= \mathrm{Var}(\beta(X_1, Y)) + \mathrm{Var}(\Theta) - 2\,\mathrm{Cov}(\beta(X_1, Y), \Theta).\end{aligned}$$

Using $\beta(X_1, Y) = \alpha + \gamma X_1 + \delta Y$ yields the result. □

Next, we characterize the dispersion around the average action.

Claim 2. Assume that $\beta(x,y) = \alpha + \gamma x + \delta y$. Then

$$\mathrm{E}\left[\left(\beta(X_1, Y) - \frac{1}{n}\sum_{j=1}^{n}\beta(X_j, Y)\right)^2\right] = \frac{n-1}{n}\gamma^2\left(\mathrm{Var}(X_1) - \mathrm{Cov}(X_1, X_2)\right).$$

Proof. Since $\mathrm{E}[\beta(X_1, Y)] = \mathrm{E}[\frac{1}{n}\sum_{j=1}^{n}\beta(X_j, Y)]$,

$$\mathrm{E}[(\beta(X_1, Y) - \frac{1}{n}\sum_{j=1}^{n}\beta(X_j, Y))^2] = \mathrm{Var}\left(\beta(X_1, Y) - \frac{1}{n}\sum_{j=1}^{n}\beta(X_j, Y)\right).$$

Since $\beta(X_j, Y) = \alpha + \gamma X_j + \delta Y$,

$$\beta(X_1, Y) - \frac{1}{n}\sum_{j=1}^{n}\beta(X_j, Y) = \gamma\left(X_1 - \frac{1}{n}\sum_{j=1}^{n}X_j\right).$$

Therefore, letting $\bar{X} = \frac{1}{n}\sum_{j=1}^{n}X_j$ and $\bar{\beta} = \frac{1}{n}\sum_{j=1}^{n}\beta(X_j, Y)$, we have that

$$\begin{aligned}\mathrm{E}[(\beta(X_1, Y) - \bar{\beta}] &= \gamma^2\mathrm{Var}(X_1 - \bar{X}) \\ &= \gamma^2\left(\mathrm{Var}(X_1) + \mathrm{Var}(\bar{X}) - 2\mathrm{Cov}(X_1, \bar{X})\right). \qquad (10)\end{aligned}$$

Using exchangeability,

$$\begin{aligned}\mathrm{Var}(\bar{X}) &= \frac{1}{n^2}\left(n\mathrm{Var}(X_1) + 2\frac{n(n-1)}{2}\mathrm{Cov}(X_1, X_2)\right) \\ &= \frac{1}{n}\mathrm{Var}(X_1) + \frac{n-1}{n}\mathrm{Cov}(X_1, X_2)\end{aligned}$$

Similarly,

$$\mathrm{Cov}(X_1, \bar{X}) = \frac{1}{n}\mathrm{Var}(X_1) + \frac{n-1}{n}\mathrm{Cov}(X_1, X_2)$$

Substituting the above two expressions in (10) gives the desired expression. □

As we can see, dispersion depends only on the private signal component of the strategy.

3.2 Binomial Signals with Uniform Prior

We now illustrate how the affine equilibrium structure arises in a canonical Bayesian model, and provide explicit expressions for equilibrium behavior and payoffs. Assume that nature chooses a value $\theta \in [0, 1]$, then a public signal is generated, and finally each player receives a private signal. Formally, let the value Θ be uniformly distributed on $[0, 1]$. Conditional on θ, let the public signal Y be distributed according to a binomial distribution with sample size L, and let the private signals be identically and independently distributed according to a binomial distribution with sample size M. Namely,

$$\Theta \sim U[0,1], \quad (Y \mid \Theta = \theta) \sim \mathrm{Bin}(L, \theta), \quad (X_i \mid \Theta = \theta) \sim \mathrm{Bin}(M, \theta), \quad i = 1, \ldots, n, \tag{11}$$

and conditional on Θ, the random variables $Y, X_1, \ldots, X_n$ are independent.

In this case the set of possible private signals is $\mathcal{X} = \{0, 1, \ldots, M\}$ and the set of possible public signals is $\mathcal{Y} = \{0, 1, \ldots, L\}$. Clearly, the higher are the sample sizes M and L, the more informative are the private and public signals about the fundamental.

We are interested in the symmetric equilibrium strategy and the equilibrium expected payoff as functions of the sample sizes M and L. Standard properties of the Binomial and Uniform distributions imply that

$$\mathrm{E}[X_i \mid \Theta = \theta] = M\theta, \qquad \mathrm{Var}(X_i \mid \Theta = \theta) = M\theta(1-\theta), \qquad i = 1, \ldots, n, \tag{12}$$

$$\mathrm{E}[Y \mid \Theta = \theta] = L\theta, \qquad \mathrm{Var}(Y \mid \Theta = \theta) = L\theta(1-\theta), \tag{13}$$

$$\mathrm{E}[\Theta] = \frac{1}{2}, \qquad \mathrm{Var}(\Theta) = \frac{1}{12}, \tag{14}$$

$$\mathrm{E}[X_i] = \mathrm{E}[\mathrm{E}[X_i \mid \Theta]] = \mathrm{E}[M\Theta] = \frac{M}{2}, \qquad i = 1, \ldots, n, \tag{15}$$

$$\mathrm{E}[Y] = \mathrm{E}[\mathrm{E}[Y \mid \Theta]] = \mathrm{E}[L\Theta] = \frac{L}{2}. \tag{16}$$

In order to compute the equilibrium strategy, we first calculate $m(x,y)$ and $\mathrm{E}[X_j \mid X_i = x, Y = y]$. By Bayes' rule, the probability density function of $(\Theta | X = x, Y = y)$ is given by $f(\theta \mid x, y) = c\theta^{x+y}(1-\theta)^{M+L-x-y}$, where c is a constant of proportionality. This means that $(\Theta | X = x, Y = y)$ follows a Beta distribution with shape parameters $x+y+1$ and $M+L-x-y$.[2] Therefore,

$$m(x,y) = \mathrm{E}[\Theta \mid X_i = x, Y = y] = \frac{1+x+y}{M+L+2}, \qquad i = 1, \dots, n. \tag{17}$$

Also, by the law of iterated expectations and the conditional independence of the signals,

$$\begin{aligned}
\mathrm{E}[X_j \mid X_i = x, Y = y] &= \mathrm{E}[\mathrm{E}[X_j \mid X_i = x, Y = y, \Theta]] \\
&= \mathrm{E}[\mathrm{E}[X_j \mid \Theta] \mid X_i = x, Y = y] \\
&= \mathrm{E}[M\Theta \mid X_i = x, Y = y] \qquad \text{by (12)} \\
&= M\frac{1+x+y}{M+L+2} \qquad \text{by (17).}
\end{aligned} \tag{18}$$

We see that the conditions for an affine equilibrium are satisfied. Hence the equilibrium strategy is affine in (x,y). We can then apply the earlier results to find the equilibrium strategy.

Proposition 3. In the uniform-binomial model, the unique symmetric equilibrium strategy is affine and given by

$$\beta(x,y) = \frac{1+bx+y}{L+2+bM}.$$

Proof. By (17) and (18), we have that

$$m(x,y) = \delta_m + \eta_m x + \zeta_m y \qquad \text{and} \qquad \mathrm{E}[X_j \mid X_i = x, Y = y] = \delta_E + \eta_E x + \zeta_E y,$$

where

$$\delta_m = \eta_m = \zeta_m = \frac{1}{M+L+2}, \qquad \delta_E = \eta_E = \zeta_E = \frac{M}{M+L+2}.$$

[2]This exploits the fact that the uniform distribution is an instance of the beta distribution (Beta(1,1)) and since the Beta distribution is the conjugate prior of the Binomial distribution, the posterior remains Beta.

Then, by Corollary 1, we obtain

$$\gamma = \frac{b}{L+2+bM}, \qquad \delta = \frac{1}{L+2+bM}, \qquad \alpha = \frac{1}{L+2+bM},$$

and

$$\beta(x,y) = \alpha + \gamma x + \delta y = \frac{1+bx+y}{L+2+bM}.$$

□

Note that the equilibrium strategy can be written as

$$\beta(x,y) = \frac{2}{L+2+bM}\cdot\frac{1}{2} + \frac{bM}{L+2+bM}\cdot\frac{x}{M} + \frac{L}{L+2+bM}\cdot\frac{y}{L}.$$

Thus, the equilibrium action is a weighted average of three components: the prior mean, the private sample proportion, and the public sample proportion. The weights depend on both the informativeness of the signals and the strategic parameter b.

The next result will allow us to calculate the players' expected equilibrium payoffs.

Claim 3. The equilibrium mean squared error relative to the fundamental is

$$\mathrm{E}[(\beta(X_1,Y)-\Theta)^2] = \frac{L+Mb^2+2}{6(L+2+bM)^2}, \tag{19}$$

and the equilibrium dispersion around the average action is

$$\mathrm{E}\left[\left(\beta(X_1,Y) - \frac{1}{n}\sum_{j=1}^{n}\beta(X_j,Y)\right)^2\right] = \frac{n-1}{n}\,\frac{Mb^2}{6(L+2+bM)^2}. \tag{20}$$

The proof consist in substituting the affine form into the general formulas derived earlier in Claims 1 and 2, and can be found in the appendix.

We can see that the mean squared error is decreasing in both M and L. The dispersion around the average action is decreasing in L, and also decreases in M. Thus, in the uniform-binomial model, more informative private and public signals improve both accuracy and coordination: the mean squared error decreases, and dispersion around the average action shrinks. As a result, the players' equilibrium expected payoff increases.

3.3 A Parametric Mixture Signal Structure

We now consider a parametric signal structure that allows us to study the effect of informativeness on equilibrium outcomes. In contrast to the previous example, increased informativeness generates a tradeoff between accuracy and coordination: more informative signals reduce the mean squared error relative to the fundamental, but increase dispersion around the average action. That is, informativeness increases accuracy, but decreases coordination. Nevertheless, increased informativeness increases the overall equilibrium payoff. For simplicity we assume there is no public signal. The signal structure is as follows. The fundamental θ takes values 0 and 1, each with equal probability, and signals take values in $X = [0, 1]$. Let $s \in [0, 2]$. Conditional on the state, signals are independent with densities

$$f_1^{(s)}(x) = 1 + s\left(x - \frac{1}{2}\right), \qquad f_0^{(s)}(x) = 1 - s\left(x - \frac{1}{2}\right), \qquad s \in [0, 2].$$

When $s = 0$, the two densities coincide and signals are completely uninformative. The informativeness of the signal structure increases with the parameter s. Indeed, note that for $0 \leq s < s' \leq 2$,

$$f_\theta^{(s)}(x, s) = \frac{s}{s'} f_\theta^{(s)}(x, s') + (1 - \frac{s}{s'}) \cdot 1.$$

Thus, the signal with parameter s is obtained by garbling the signal with parameter s' with the uniform distribution, namely the latter Blackwell dominates the former.

We are interested in the unique equilibrium strategy identified in Theorem 1. Since $f_1^{(s)}(x) + f_0^{(s)}(x) = 2$, Bayes' rule gives

$$P(\theta = 1 | X_i = x) = \frac{f_1^{(s)}(x)}{2} \text{ and } P(\theta = 0 \mid X_i = x) = \frac{f_0^{(s)}(x)}{2}.$$

Hence $m(x) = \mathrm{E}[\theta \mid X_i = x] = \frac{2-s}{4} + \frac{s}{2}x$. A direct calculation gives

$$\mathrm{E}[X_i \mid \theta = 1] = \frac{1}{2} + \frac{s}{12}, \qquad \mathrm{E}[X_i \mid \theta = 0] = \frac{1}{2} - \frac{s}{12}. \tag{21}$$

Thus,

$$\begin{aligned}
\mathrm{E}[X_j \mid X_i = x] &= \mathrm{E}[\mathrm{E}[X_j \mid \theta] \mid X_i = x] \\
&= P(\theta = 1 \mid X_i = x)\mathrm{E}[X_i \mid \theta = 1] + P(\theta = 0 \mid X_i = x)\mathrm{E}[X_i \mid \theta = 0] \\
&= \frac{1}{24}(12 - s^2) + \frac{s^2}{12}x.
\end{aligned}$$

Note that $m(x)$ and $\mathrm{E}[X_j \mid X_i = x]$ are affine functions. Furthermore, using (21), we can see that $\mathrm{E}[X_i] = 1/2 = \mathrm{E}[\Theta]$. Then, by Corollary 2,

$$\beta(x) = \frac{1-\gamma}{2} + \gamma x$$

where by Corollary 1 $\gamma = \frac{6bs}{12-(1-b)s^2}$. We see that the more informative the signal is, the more responsive the equilibrium strategy is to it.

Concerning the equilibrium payoffs, we have the following.

Claim 4. The equilibrium mean square error relative to the fundamental is

$$\mathrm{E}[(\beta(X_1) - \theta)^2] = \frac{3b^2s^2}{\left((b-1)s^2+12\right)^2} - \frac{bs^2}{2\left((b-1)s^2+12\right)} + \frac{1}{4} \tag{22}$$

and the equilibrium dispersion around the average action is

$$\mathrm{E}[(\beta(X_1) - \bar{\beta})^2] = \frac{n-1}{n}\frac{b^2s^2\left(12-s^2\right)}{4(12-(1-b)s^2)^2}. \tag{23}$$

The proof consists in substituting the affine equilibrium into the general expressions derived earlier in Claims 1 and 2, and can be found in the appendix.

It can be checked that, for $s > 0$, the mean squared error is decreasing in the informativeness parameter, and that the dispersion around the average action is increasing in the informativeness parameter. However, it can also be checked that their sum is decreasing in s. Thus, equilibrium payoffs increase with the informativeness of private signals.

4 A finite mixture signal structure model

In this section we analyze a general information structure that leads to equilibria that are not necessarily affine. They are, however, linear in the posteriors about the fundamental, and can be computed by solving a linear system of equations. Consider a finite mixture signal structure in which nature chooses the value of the fundamental $\theta \in \mathcal{S}$, and conditional on θ, a public signal $y \in \mathcal{Y}$ and private signals $x_i \in \mathcal{X}$ are drawn independently, conditional on θ, from continuous densities $g_\theta(y)$, and $f_\theta(x_i)$, respectively. For simplicity, we assume that $\mathcal{S}$ is finite, and denote by $\pi_\theta > 0$ the probability of θ. We further assume that $\sum_{\theta \in \mathcal{S}} g_\theta(y) f_\theta(x) > 0$ for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ so that the posterior probabilities are well-defined for all (x, y). Denote the conditional probability that the realized fundamental is θ given the private signal x and the public signal y, by

$$w_\theta(x, y) = \Pr(\Theta = \theta \mid X_i = x, Y = y) = \frac{\pi_\theta g_\theta(y) f_\theta(x)}{\sum_{\theta' \in \mathcal{S}} \pi_{\theta'} g_{\theta'}(y) f_{\theta'}(x)}$$

Notice that because X_j and Y are conditionally independent given Θ, the distribution of X_j given $\Theta = \theta, Y = y$ is f_θ. Thus, conditional on $\Theta = \theta$, X_j is distributed according to f_θ, and therefore $\mathrm{E}[w_{\theta'}(X_j, y) \mid \Theta = \theta] = \int_{\mathcal{X}} w_{\theta'}(x, y) f_\theta(x)\, dx$. For future reference, denote

$$M_{\theta,\theta'}(y) = \mathrm{E}[w_{\theta'}(X_j, y) \mid \Theta = \theta] \tag{24}$$

and let $M(y)$ be the $|S| \times |S|$ matrix whose (θ, θ') entry is $M_{\theta,\theta'}(y)$. Note that for any fixed y, $M(y)$ is a stochastic matrix.

Let $\mathbf{v} = (\theta)_{\theta \in \mathcal{S}}$ be the column vector of fundamental states, and let $\mathbf{w}(x, y) = (w_\theta(x, y))_{\theta \in \mathcal{S}}$ be the row vector of posterior probabilities given the observed signals. Note that its entries add up to one. Then,

$$m(x, y) = \mathrm{E}[\Theta \mid X_i = x, Y = y] = \sum_{\theta \in \mathcal{S}} \theta w_\theta(x, y) = \mathbf{w}(x, y) \cdot \mathbf{v} \tag{25}$$

and the equilibrium strategy, for a fixed public signal y, solves the functional equation

$$\beta(x, y) = (1 - b)(K\beta)(x, y) + b\mathbf{w}(x, y) \cdot \mathbf{v} \tag{26}$$

where K is defined by $(K\phi)(x, y) = \mathrm{E}[\phi(X_j, Y) \mid X_i = x, Y = y]$.

Proposition 4. In a finite mixture signal structure model, the equilibrium strategy is linear in posterior beliefs. That is, for each $y \in \mathcal{Y}$, there exists a vector $\mathbf{c}(y) \in \mathbb{R}^{|S|}$ such that

$$\beta(x, y) = \mathbf{w}(x, y) \cdot \mathbf{c}(y).$$

Proof. Let $\mathcal{B}'$ be the set of functions $\phi : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ such that, for each y, there exists $\mathbf{c}(y) \in \mathbb{R}^{|S|}$ with $\phi(x, y) = \mathbf{w}(x, y) \cdot \mathbf{c}(y)$. It is a finite-dimensional vector space. Fix $\phi \in \mathcal{B}'$, so that $\phi(x, y) = \mathbf{w}(x, y) \cdot \mathbf{c}(y)$ for some $\mathbf{c}(y) \in \mathbb{R}^{|S|}$. Then,

$$(K\phi)(x, y) = \mathrm{E}[\mathbf{w}(X_j, y) \cdot \mathbf{c}(y) \mid X_i = x, Y = y]$$

By the law of iterated expectations and the conditional independence of the signals given the state Θ,

$$\begin{aligned} (K\phi)(x, y) &= \sum_{\theta \in S} \Pr(\Theta = \theta \mid X_i = x, Y = y) \mathrm{E}[\mathbf{w}(X_j, y) \cdot \mathbf{c}(y) \mid \Theta = \theta, Y = y] \\ &= \sum_{\theta \in S} w_\theta(x, y) \mathrm{E}[\mathbf{w}(X_j, y) \mid \Theta = \theta] \cdot \mathbf{c}(y) \end{aligned}$$

Using the definition of M (see equation (24)), we have that

$$(K\phi)(x, y) = \mathbf{w}(x, y) \cdot M(y)\mathbf{c}(y)$$

which shows that $K\phi$ belongs to $\mathcal{B}'$. Also, by equality (25), m belongs to $\mathcal{B}'$ as well. Therefore, by Observation 1, the equilibrium strategy $\beta \in \mathcal{B}'$. That is, there is $\mathbf{c}(y) = (c_\theta(y))_{\theta \in S}$ such that $\beta(x, y) = \mathbf{w}(x, y) \cdot \mathbf{c}(y)$. ☐

The previous result shows that equilibrium strategies depend on signals only through posterior beliefs. We now exploit this structure to obtain an explicit characterization in the finite-state case.

Corollary 3. In a finite mixture signal structure model, the unique symmetric equilibrium strategy conditional on the public signal is

$$\beta(x, y) = b\mathbf{w}(x, y) \cdot (I - (1 - b)M(y))^{-1}\mathbf{v}$$

Proof. For each $y \in \mathcal{Y}$, let $M(y)$ be the $|\mathcal{S}| \times |\mathcal{S}|$ matrix defined in equation (24). The equilibrium strategy β is the unique solution to the functional equation (26), which by Proposition 4, can be written as

$$\mathbf{w}(x, y) \cdot \mathbf{c}(y) = \mathbf{w}(x, y) \cdot ((1 - b)M(y)\mathbf{c}(y) + b\mathbf{v}) \quad \forall (x, y) \in \mathcal{X} \times \mathcal{Y}$$

For any given y, this equation holds for all x if $\mathbf{c}(y) = ((1 - b)M(y)\mathbf{c}(y) + b\mathbf{v})$, or, rearranging, if

$$(I - (1 - b)M(y))\mathbf{c}(y) = b\mathbf{v}$$

Because $M(y)$ is a stochastic matrix and $b \in (0, 1)$, the matrix $(I - (1 - b)M(y))$ is invertible. Solving for the vector $\mathbf{c}(y)$,

$$\mathbf{c}(y) = b(I - (1 - b)M(y))^{-1}\mathbf{v}$$

By Proposition 4, we obtain the desired result. □

We end with an example that illustrates how the finite mixture structure naturally generates bounded, non-affine equilibrium. strategies

Example 1. Consider a simple setting with two equiprobable fundamental states, $\mathbf{v} = (1, 2)'$. For simplicity, assume that there is no public signal, or equivalently, that the public signal is uninformative. Suppose the signal distributions conditional on the state are given by piecewise linear densities on the interval $[0, 1]$:

$$f_1(x) = 3\min(2x, 1 - x)$$
$$f_2(x) = 3\min(x, 2(1 - x))$$

These densities represent a scenario where state 1 makes lower signals more likely, while state 2 makes higher signals more likely. The posterior weights are $w_1(x) = \frac{f_1(x)}{f_1(x)+f_2(x)}$ and $w_2(x) = 1 - w_1(x)$, and therefore the expectation matrix M, where $M_{\theta,\theta'} = \mathrm{E}_{f_\theta}[w_{\theta'}(X_i)]$ is given by

$$M = \begin{pmatrix} 29/54 & 25/54 \\ 25/54 & 29/54 \end{pmatrix}$$

Applying the closed-form solution derived above, $\beta(x) = bw(x)(I - (1-b)M)^{-1}\mathbf{v}$, we obtain the following continuous equilibrium strategy:

$$\beta(x) = \begin{cases} \frac{75-3b}{50+4b} & \text{if } x < 1/3 \\ \frac{75-21b+54bx}{50+4b} & \text{if } 1/3 \le x \le 2/3 \\ \frac{15(5+b)}{50+4b} & \text{if } x > 2/3 \end{cases}$$

This example highlights two key features. First, equilibrium strategies need not be affine; here they are piecewise linear. Second, for extreme signal realizations, posterior beliefs become saturated, so marginal changes in signals do not affect beliefs. As a result, equilibrium actions are locally insensitive to signals in the tails.

5 Appendix

Proof of Claim 3: Recall that by the law of total covariance, for any random variables W, Y, Z,

$$\mathrm{Cov}(Y, Z) = \mathrm{E}[\mathrm{Cov}(Y, Z) \mid W] + \mathrm{Cov}(\mathrm{E}[Y \mid W], \mathrm{E}[Z \mid W]). \tag{27}$$

In particular, when $Y = Z$, and $W = \Theta$,

$$\mathrm{Var}(Y) = \mathrm{E}[\mathrm{Var}(Y) \mid W] + \mathrm{Var}(\mathrm{E}[Y \mid W]).$$

Consequently,

$$\begin{aligned} \mathrm{Var}(X_i) &= \mathrm{E}[\mathrm{Var}(X_i \mid \Theta)] + \mathrm{Var}(\mathrm{E}[X_i \mid \Theta]) \\ &= \mathrm{E}[M\Theta(1-\Theta)] + \mathrm{Var}(M\Theta) \qquad \text{by (12)–(13)} \\ &= \frac{M}{6} + \frac{M^2}{12} = \frac{M(M+2)}{12}, \end{aligned} \tag{28}$$

and similarly,

$$\mathrm{Var}(Y) = \frac{L(L+2)}{12}. \tag{29}$$

Applying (27) to X_i and X_j, $i \neq j$, we have

$$\begin{aligned}
\mathrm{Cov}(X_i, X_j) &= \mathrm{E}[\mathrm{Cov}(X_i, X_j)|\Theta] + \mathrm{Cov}(\mathrm{E}[X_i|\Theta], \mathrm{E}[X_j|\Theta]) \\
&= \mathrm{Cov}(\mathrm{E}[X_1|\Theta], \mathrm{E}[X_2|\Theta]) \qquad \text{by conditional independence of } X_i \text{ and } X_j \\
&= \mathrm{Cov}(M\Theta, M\Theta) \qquad \text{by (12)} \\
&= \mathrm{Var}(M\Theta) = \frac{M^2}{12} \qquad \text{by (14)} \qquad (30)
\end{aligned}$$

and, similarly

$$\mathrm{Cov}(X_i, Y) = \mathrm{Cov}(M\Theta, L\Theta) = \frac{ML}{12}. \qquad (31)$$

Also, note that

$$\mathrm{E}[X_1\Theta] = \mathrm{E}[\Theta\, \mathrm{E}[X_1 \mid \Theta]] = \mathrm{E}[M\Theta^2] = \frac{M}{3}$$

Therefore, since $\mathrm{Cov}(X_1, \Theta) = \mathrm{E}[X_1\Theta] - \mathrm{E}[X_1]\mathrm{E}[\Theta]$, using (14-15) we have that

$$\mathrm{Cov}(X_1, \Theta) = \frac{M}{3} - \frac{M}{2}\frac{1}{2} = \frac{M}{12}. \qquad (32)$$

Similarly,

$$\mathrm{Cov}(Y, \Theta) = \frac{L}{3} - \frac{L}{2}\frac{1}{2} = \frac{L}{12}. \qquad (33)$$

By Claim 1,

$$\begin{aligned}
\mathrm{E}[(\beta(X_1, Y) - \Theta)^2] = \gamma^2 \mathrm{Var}(X_1) + \delta^2 \mathrm{Var}(Y) + 2\gamma\delta\, \mathrm{Cov}(X_1, Y) + \mathrm{Var}(\Theta) \\
- 2\gamma\, \mathrm{Cov}(X_1, \Theta) - 2\delta\, \mathrm{Cov}(Y, \Theta). \qquad (34)
\end{aligned}$$

Substituting (28), (29), (31), (14), (32), (33), and (16) into (34), and given that

$$\gamma = \frac{b}{L + 2 + bM}, \qquad \delta = \frac{1}{L + 2 + bM}, \qquad \alpha = \frac{1}{L + 2 + bM},$$

simplifying, we obtain (19).

For the dispersion term, by Claim 2, and using (28) and (30).

$$\begin{aligned}
\mathrm{E}\left[\left(\beta(X_1, Y) - \frac{1}{n}\sum_{j=1}^{n} \beta(X_j, Y)\right)^2\right] &= \frac{n-1}{n}\gamma^2 \left(\mathrm{Var}(X_1) - \mathrm{Cov}(X_1, X_2)\right) \\
&= \frac{n-1}{n}\gamma^2 \left(\frac{M(M+2)}{12} - \frac{M^2}{12}\right) \\
&= \frac{n-1}{n}\gamma^2 \frac{M}{6}.
\end{aligned}$$

Using $\gamma = \frac{b}{L+2+bM}$ yields (20). □

Proof of Claim 4: By direct calculation,

$$\mathrm{E}[X_i] = \frac{1}{2}, \qquad Var(X_i) = \frac{1}{12}, \qquad Cov(X_i, \theta) = \frac{s}{24}, \qquad Var(\theta) = \frac{1}{4}.$$

By Claim 1

$$\mathrm{E}[(\beta(X_1) - \Theta)^2] = \gamma^2 \mathrm{Var}\,(X_1) + \mathrm{Var}(\theta) - 2\gamma\left(\mathrm{E}[X_1\Theta] - \mathrm{E}[X_1]\mathrm{E}[\Theta]\right).$$

Since

$$\begin{aligned}
\mathrm{E}[X_1\Theta] &= \mathrm{E}[\Theta \mathrm{E}[X_1|\Theta]] = \frac{1\mathrm{E}[X_1|\Theta=1] + 0\mathrm{E}[X_1|\Theta=0]}{2} \\
&= \mathrm{E}[X_1|\Theta=1]/2 \\
&= \frac{1}{4} + \frac{s}{24}
\end{aligned}$$

we get

$$\begin{aligned}
\mathrm{E}[(\beta(X_1) - \theta)^2] &= \gamma^2 \frac{1}{12} + \frac{1}{4} - 2\gamma\left(\frac{1}{4} + \frac{s}{24} - \frac{1}{2}\frac{1}{2}\right) \\
&= \gamma^2 \frac{1}{12} + \frac{1}{4} - 2\gamma\frac{s}{24}.
\end{aligned}$$

Substituting $\gamma = \frac{6bs}{12-(1-b)s^2}$ and simplifying gives equality (22). As for equality (23), let

$$\bar{\beta} = \frac{1}{n}\sum_{j=1}^{n} \beta(X_j).$$

By Claim (2),

$$\mathrm{E}[(\beta(X_1) - \bar{\beta})^2] = \frac{n-1}{n}\gamma^2\left(\mathrm{Var}\,(X_1) - \mathrm{Cov}\,(X_1, X_2)\right)$$

Since

$$\begin{aligned}
Cov(X_1, X_2) &= Var(\mathrm{E}[X_1 \mid \theta]) \\
&= \frac{\left(\mathrm{E}[X_i \mid \theta = 1] - \frac{1}{2}\right)^2 + \left(\mathrm{E}[X_i \mid \theta = 0] - \frac{1}{2}\right)^2}{2} \\
&= \frac{\left(\frac{s}{12}\right)^2 + \left(\frac{s}{12}\right)^2}{2} \\
&= \frac{s^2}{144},
\end{aligned}$$

we have

$$\mathrm{E}[(\beta(X_1) - \bar{\beta})^2] = \frac{n-1}{n}\gamma^2 \left(\mathrm{Var}\,(X_1) - \mathrm{Cov}\,(X_1, X_2)\right)$$
$$= \frac{n-1}{n}\gamma^2 \left(\frac{1}{12} - \frac{s^2}{144}\right).$$

Substituting $\gamma = \frac{6bs}{12-(1-b)s^2}$, we obtain (23). □

6 Declaration of generative AI and AI-assisted technologies in the manuscript preparation process

During the preparation of this work the author used ChatGPT (OpenAI) in order to assist with improving the clarity of exposition, refining mathematical arguments, checking consistency of notation, and editing the language of the manuscript. After using this tool, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

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